TRACE MAPS FOR QUARTERNARY
QUASIPERIODIC LATTICES

KAZUMOTO IGUCHI
Laboratory for Computational Sciences, Fujitsu Limited,
1-17-25 Shinkamata, Ota-ku, Tokyo 144, Japan

Received 16 December 1992

We derive trace maps for the quarternary quasiperiodic lattices in one dimension, which are constructed by four letters, a, b, c, and d. We prove that the dimension of the trace map is ten.

Since the discovery of quasicrystals and their one-dimensional modeling, Schrödinger operators with an arbitrary deterministic potential sequence have been of interest. The sequences are classified in between periodic and random sequences. There are many such systems of two letters such as the Fibonacci chain and its generalized ones and the Thue–Morse sequence. There are more generic systems such as the Rudin–Shapiro sequence, the Circle sequence, and the ternary quasiperiodic sequences. And there appeared generalizations to the systems with an arbitrary number of letters. However, the explicit forms of the trace map in the systems of many letters have not been well understood until recently.

In this letter we would like to restrict ourselves to the system of the quarternary quasiperiodic lattices constructed by four letters: a, b, c, and d, in order to know what is going on in our problem. And we shall explicitly derive its trace map.

Denote the species of atoms by a set of four letters

\[ \Lambda_4 \equiv \{ a, b, c, d \} \]  

(1)

This set forms a basis for construction of a quarternary quasiperiodic lattice. We remark the following. Physicists are concerned with the one-dimensional lattice problem to understand its physical properties. In this case, the letters are thought of as the \( 2 \times 2 \) matrices belonging to \( G \equiv SL(2, \mathbb{C}) \) defined by the physical models such as the Schrödinger equation, the Maxwell equation, etc. For this purpose we denote the matrix-valued letters by its capital letters \( A, B, C, \) and \( D \) corresponding to the four letters \( a, b, c, \) and \( d \). And a string of letters (i.e., a word) is regarded as the unit cell for an infinite chain, say \( L(a, b, c, d) \).

PACS Nos.: 02.20.-b, 71.10.+x, 73.20.Dx.
The inflation-deflation (i.e., the scaling transformation or the renormalization group transformation) can be thought of as a substitution scheme of letters: If four strings of words, \( W_1(a, b, c, d) \), \( W_2(a, b, c, d) \), \( W_3(a, b, c, d) \), and \( W_4(a, b, c, d) \) are set, then a substitution scheme \( S \) that inflates (or deflates) the lattice from one generation to another is defined as follows:

\[
S = \begin{cases}
   a \rightarrow W_1(a, b, c, d) \\
   b \rightarrow W_2(a, b, c, d) \\
   c \rightarrow W_3(a, b, c, d) \\
   d \rightarrow W_4(a, b, c, d).
\end{cases}
\]  

(2)

Mathematically speaking, this type of transformations is equivalent to a generator of an automorphism on the free group \( F_4 \) constructed by \( \Lambda_4 \), where

\[
F_4 = \langle a, b, c, d \rangle.
\]  

(3)

There is a profound mathematical theorem due to Nielsen:

Nielsen's Theorem: Any substitution is generated by a combination of the five classes of generators: inversion (\( J \)), exchange (\( X \)), cyclic permutation (\( P \)), left and right multiplications (\( L \) and \( R \)), where we have

\[
J^2 = X^2 = P^4 = 1 \text{ (identity)}. 
\]  

(4)

For the quaternary quasiperiodic lattices, there are four generators in \( J \) (i.e., \( J_1, J_2, J_3, \) and \( J_4 \)), six generators in \( X \) (i.e., \( X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, \) and \( X_{34} \)), \( L \) (i.e., \( L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, \) and \( L_{34} \)) and \( R \) (i.e., \( R_{12}, R_{13}, R_{14}, R_{23}, R_{24}, \) and \( R_{34} \)), respectively. And there are two generators in \( P \) (i.e., \( P \) and \( P^{-1} \)). For example, we have

\[
J_1 = \begin{cases}
   a \rightarrow a^{-1} \\
   b \rightarrow b \\
   c \rightarrow c \\
   d \rightarrow d.
\end{cases}
\]  

(5)

\[
X_{12} = \begin{cases}
   a \rightarrow b \\
   b \rightarrow a \\
   c \rightarrow c \\
   d \rightarrow d.
\end{cases}
\]  

(6)

\[
L_{12} = \begin{cases}
   a \rightarrow b \\
   b \rightarrow b \\
   c \rightarrow c \\
   d \rightarrow d.
\end{cases}
\]  

(7)

\[
R_{12} = \begin{cases}
   a \rightarrow b \\
   b \rightarrow b \\
   c \rightarrow c \\
   d \rightarrow d.
\end{cases}
\]  

(8)

and

\[
P = \begin{cases}
   a \rightarrow d \\
   b \rightarrow a \\
   c \rightarrow b \\
   d \rightarrow c.
\end{cases}
\]  

(9)
From this theorem any substitution $S_k$ can be decomposed into a string of the generators in the five classes $J$, $X$, $P$, $L$, and $R$. We call a free group generated by them $N_4$. For example, consider the system of inflation:

$$d \rightarrow c \rightarrow b \rightarrow a \rightarrow abcd \rightarrow \ldots.$$  \hspace{1cm} (10)

If we set the generations of the unit cells as

$$L_0(a, b, c, d) \equiv d,$$
$$L_1(a, b, c, d) \equiv c,$$
$$L_2(a, b, c, d) \equiv b,$$
$$L_3(a, b, c, d) \equiv a,$$
$$L_4(a, b, c, d) \equiv abcd,$$  \hspace{1cm} (11)

and so forth, then the scaling transformation is described by

$$S_k : U_k \equiv (L_k, L_{k+1}, L_{k+2}, L_{k+3}) \rightarrow U_{k+1} \equiv (L_{k+1}, L_{k+2}, L_{k+3}, L_{k+4}),$$

equivalently

$$U_{k+1} = S_k U_k,$$  \hspace{1cm} (12)

where

$$S_k \equiv PR_{14}R_{24}R_{34}. $$  \hspace{1cm} (13)

Similarly for the sequence of inflation

$$d \rightarrow c \rightarrow b \rightarrow a \rightarrow a^{m_1}b^{m_2}c^{m_3}d \rightarrow \ldots,$$  \hspace{1cm} (14)

we get

$$S_k \equiv PR_{14}^{b_1}R_{24}^{m_2}R_{34}^{m_3},$$  \hspace{1cm} (15)

where $m_1$, $m_2$, $m_3$ are positive integers.

Let $A_j$, $B_j$, $C_j$, $D_j$ be the total numbers of $a$, $b$, $c$, $d$ in the $j$th unit cell $L_j(a, b, c, d)$, respectively. Then, the total number of letters in the unit cell $N_j$ is obviously given by

$$N_j \equiv A_j + B_j + C_j + D_j.$$  \hspace{1cm} (16)

The development of the total number of letters $N_j$ is represented by

$$N_{j+4} = N_{j+3} + N_{j+2} + N_{j+1} + N_j,$$  \hspace{1cm} (17)

for Eq. (10) and

$$N_{j+4} = m_1 N_{j+3} + m_2 N_{j+2} + m_3 N_{j+1} + N_j,$$  \hspace{1cm} (18)
for Eq. (14) with the initial condition

\[ N_3 = N_2 = N_1 = N_0 = 1. \]  

(19)

The ratio \( N_j+1/N_j \) can converge to an irrational number that is one of the roots of an algebraic equation associated with Eq. (17) or (18). In this way we can recognize that the ratio is generalized version of the continued fraction expansion for the systems of two letters to our problem of four letters.\textsuperscript{4,11}

We are going to derive the trace map for an arbitrary quaternary quasiperiodic sequence generated by \( \Lambda_4 \). Denote the four traces by \( t_1 \equiv \text{tr} A, \ t_2 \equiv \text{tr} B, \ t_3 \equiv \text{tr} C, \ t_4 \equiv \text{tr} D \). And also denote the trace of \( AB \) by \( t_{12} \equiv \text{tr}(AB) \), and likewise, we define as \( t_{123} \equiv \text{tr}(ABC) \), and so forth. We note some symmetric properties of traces:

\[
\begin{align*}
  t_1 &= \text{tr}(A^{-1}) = t_1; \text{ inversion symmetry, } \\
  t_{12} &= t_{21}; \text{ exchange symmetry, } \\
  t_{123} &= t_{231} = t_{312}; \text{ cyclic permutation symmetry, }
\end{align*}
\]

(20)

etc. Since there exist some identities that we sometimes call the Fricke identities:\textsuperscript{17}

\[
\text{tr}(A^{-1}B) = \text{tr} A \cdot \text{tr} B - \text{tr}(AB)
\]

and

\[
\text{tr}(A^{-1}B^{-1}AB) = (\text{tr} A)^2 + (\text{tr} B)^2 + (\text{tr}(AB))^2 - \text{tr} A \cdot \text{tr} B \cdot \text{tr}(AB) - 2,
\]

we first keep

\[
\begin{align*}
  t_{12} &\equiv \text{tr}(A^{-1}B) = t_1 t_2 - t_{12}, \\
  \Lambda_{12} &\equiv \text{tr}(A^{-1}B^{-1}AB) = t_1^2 + t_2^2 + t_{12}^2 - t_1 t_2 t_{12} - 2.
\end{align*}
\]

(21)  

(22)

The above two identities are held for any combination of the two matrices in the set of the four transfer matrices.\textsuperscript{18} Furthermore, we have the following identities:\textsuperscript{17,19,20}

\[
\begin{align*}
  p &\equiv t_{123} + t_{132} = t_1 t_{23} + t_2 t_{31} + t_3 t_{12} - t_1 t_2 t_3, \\
  q &\equiv t_{123} \cdot t_{132} = t_1^2 + t_2^2 + t_3^2 + t_{12}^2 + t_{23}^2 + t_{13}^2 \\
  &\quad + t_1 t_{23} t_{13} - t_1 t_2 t_{12} - t_2 t_3 t_{23} - t_1 t_3 t_{13} - 4.
\end{align*}
\]

(23a)  

(23b)

From these, \( t_{123} \) and \( t_{132} \) are the roots of a quadratic equation

\[
t^2 - pt + q = 0,
\]

respectively. In this way a trace of the three different matrices is represented in terms of the six traces: \( t_1, \ t_2, \ t_3, \ t_{12}, \ t_{23} \) and \( t_{13} \).
Let us consider the actions of the generators of the Nielsen transformations $N_4$ on the $4(=4C_1)$ traces

$$t_1, t_2, t_3, t_4.$$  \(25\)

We find that the left or right multiplication relates the four traces to the six $(=4C_2)$ traces of two matrices,

$$t_{12}, t_{13}, t_{14}, t_{23}, t_{24}, t_{34}. \quad 26$$

Therefore, we have to take into account the actions of the five classes of the generators on the six traces as well. This relates them to four $(=4C_3)$ traces,

$$t_{123}, t_{124}, t_{134}, t_{234}. \quad 27$$

and finally the above four traces are related to one trace of four letters

$$t_{1234}. \quad 28$$

In this way we find that $15(=2^4-1=4C_1+4C_2+4C_3+4C_4)$ traces be treated under the actions of the generators. This result is summarized in Table 1.\(^{21}\)

Table 1. The actions of the generators for the Nielsen transformations on the 15 traces. If a string of the generators in the five classes (e.g., Eq. (13)) is taken into account, then we are able to derive the trace map at each stage of the scaling transformations according to this table.

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$t_3$</th>
<th>$t_4$</th>
<th>$t_{12}$</th>
<th>$t_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$t_3$</td>
<td>$t_4$</td>
<td>$t_{12} - t_{13}$</td>
<td></td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>$t_2$</td>
<td>$t_1$</td>
<td>$t_3$</td>
<td>$t_4$</td>
<td>$t_{13}$</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>$t_4$</td>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$t_3$</td>
<td>$t_{14}$</td>
<td></td>
</tr>
<tr>
<td>$L_{12}$</td>
<td>$t_{12}$</td>
<td>$t_2$</td>
<td>$t_3$</td>
<td>$t_4$</td>
<td>$t_{12}t_2 - t_1$</td>
<td></td>
</tr>
<tr>
<td>$R_{12}$</td>
<td>$t_{12}$</td>
<td>$t_2$</td>
<td>$t_3$</td>
<td>$t_4$</td>
<td>$t_{12}t_2 - t_1$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_4$</th>
<th>$t_{23}$</th>
<th>$t_{24}$</th>
<th>$t_{34}$</th>
<th>$t_{123}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$t_1t_4 - t_{14}$</td>
<td>$t_{23}$</td>
<td>$t_{24}$</td>
<td>$t_{34}$</td>
<td>$t_{123} - t_{123}$</td>
<td></td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>$t_{24}$</td>
<td>$t_{13}$</td>
<td>$t_{14}$</td>
<td>$t_{34}$</td>
<td>$t_{132}$</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>$t_{34}$</td>
<td>$t_{12}$</td>
<td>$t_{13}$</td>
<td>$t_{23}$</td>
<td>$t_{124}$</td>
<td></td>
</tr>
<tr>
<td>$L_{12}$</td>
<td>$t_{12}$</td>
<td>$t_{14}$</td>
<td>$t_{23}$</td>
<td>$t_{24}$</td>
<td>$t_{12}t_2 - t_1$</td>
<td></td>
</tr>
<tr>
<td>$R_{12}$</td>
<td>$t_{12}$</td>
<td>$t_{23}$</td>
<td>$t_{24}$</td>
<td>$t_{34}$</td>
<td>$t_{12}t_2 - t_1$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$t_{12}$</th>
<th>$t_{134}$</th>
<th>$t_{234}$</th>
<th>$t_{1234}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$t_1t_{24} - t_{124}$</td>
<td>$t_{134} - t_{134}$</td>
<td>$t_{234}$</td>
<td>$t_{12}t_{34} - t_{1234}$</td>
</tr>
<tr>
<td>$X_{12}$</td>
<td>$t_{14}$</td>
<td>$t_{234}$</td>
<td>$t_{134}$</td>
<td>$t_{2134}$</td>
</tr>
<tr>
<td>$P$</td>
<td>$t_{13}$</td>
<td>$t_{234}$</td>
<td>$t_{124}$</td>
<td>$t_{1234}$</td>
</tr>
<tr>
<td>$L_{12}$</td>
<td>$t_{12}t_2 - t_1$</td>
<td>$t_{124}$</td>
<td>$t_{234}$</td>
<td>$t_{12}t_2 - t_{134}$</td>
</tr>
<tr>
<td>$R_{12}$</td>
<td>$t_{12}t_2 - t_1$</td>
<td>$t_{2134}$</td>
<td>$t_{234}$</td>
<td>$t_{12}t_{234} - t_{134}$</td>
</tr>
</tbody>
</table>

$t_{1234} = -t_{1234} + (t_{12} - t_1t_2)t_{34} + t_{12}t_{34} + t_2t_{134}$
We now prove that $t_{1234}$ can be represented in terms of the ten traces of Eqs. (25) and (26). Consider $t_{2134}$. From the lemma in Ref. 21, we find an identity:

$$t_{2134} = -t_{1234} + (t_{12} - t_{12})t_{34} + t_{1}t_{234} + t_{2}t_{134},$$

$$\equiv -t_{1234} + B(12; 34),$$

where the expression $B(12; 34)$ is obviously symmetric under the exchange between 1 and 2, namely

$$B(12; 34) = B(21; 34).$$

Repeating the above interchange three times, $t_{1234} = -t_{2134} + B(12; 34) = -t_{4213} + B(12; 34) = -[t_{2413} + B(24; 13)] + B(12; 34) = t_{3241} - B(24; 13) + B(12; 34) = [-t_{2341} + B(23; 41)] - B(24; 13) + B(12; 34) = -t_{1234} + B(23; 41) - B(24; 13) + B(12; 34).$ Therefore, we get

$$2 \cdot t_{1234} = B(23; 41) - B(24; 13) + B(12; 34)$$

$$= (t_{23} - t_{23}t_{14} + t_{23}t_{14} + t_{23}t_{14})$$

$$- [(t_{24} - t_{24}t_{13}) + t_{24}t_{13} + t_{4}t_{213}]$$

$$+ (t_{12} - t_{12})t_{34} + t_{1}t_{234} + t_{2}t_{134}.$$

The above identity agrees with the sixth equation in the theorem of Whittemore.21

Denote by $\Psi_0$ the initial set of all the ten traces [Eqs. (25) and (26)]. Operating the scaling transformation $S_0$, we get $\Psi_1 = S_{0}\Psi_0$. Recursively operating this, we get the trace map for our system as

$$\Psi_{k+1} = S_k \Psi_k.$$  

Since an arbitrary substitution is represented in terms of a string of the generators in the five classes (see, Eqs. (13) and 15)), the dimension of the trace map is invariant under the scaling transformation and apparently it is ten. In this way, a trace of four letters is represented in terms of the ten traces of less than three letters.

In conclusion, it is shown that the scaling transformations in the quarternary quasiperiodic lattices in one dimension is equivalent to a profound mathematical concept, the Nielsen transformations, which are automorphisms of a free group $F_4$ generated by four letters. We have shown that the automorphisms induce the trace map on the traces (which is regarded as the inner automorphisms).

Acknowledgements

The author would like to thank Y. Avishai and D. Berend for giving him a preprint of their work prior to publication.

References


16. W. Magnus, A. Karrass, and D. Solitar, Combinatorial Group Theory (Dover, 1976), p. 120.

17. R. Fricke and F. Klein, Vorlesungen uber Theorie der Automorphen Functionen (Teuber, 1897); H. Cohn, Ann. Math. 61, 1 (1955); H. Cohn, Acta Arith. 18, 125 (1971);
It is worth mentioning that Eq. (5) is equivalent to the invariant surface first found by Kohmoto, Kadanoff, and Tang. If we parametrize \( x = \frac{1}{2} t_1^2, y = \frac{1}{2} t_2^2, \) and \( z = t_1 t_2 \), then \( l_{12} \equiv (A_{12} - 2)/4 = x^2 + y^2 + z^2 - 2xyz - 1. \)

21. To prove the results in Table 1, the Fricke identities have been fully used, together with a theorem due to Horowitz: If \( u \) is an arbitrary element of \( F_n \), that is a free group constructed by \( n \) letters, then the trace of \( u \) can be expressed as a polynomial, 

\[
\text{tr} u = P(t_1, t_2, \ldots, t_n; t_{12}, t_{23}, \ldots, t_{n-1n}; t_{12}, \ldots, t_{n-2,n-1,n}; \ldots; t_{12\cdot\ldots\cdot n-1,n}; t_{12\cdot\ldots\cdot n})
\]

with integer coefficients in the \( 2^n - 1 \) traces. This is proved by using the following lemma: Take the \( s \) generators \( g_1, g_2, \ldots, g_s \) \((s < n)\) in \( F_n \). Then there is the following identity:

\[
\text{tr}(g_1 g_2 \cdots g_{s-1} g_s) = \text{tr}(g_1 g_2 \cdots g_{s-2})[\text{tr}(g_{s-1} g_s) - \text{tr}(g_s) \text{tr}(g_{s-1})] + \text{tr}(g_{s-1})[\text{tr}(g_1 g_2 \cdots g_{s-2} g_s) + \text{tr}(g_s) \text{tr}(g_1 g_2 \cdots g_{s-1}) - \text{tr}(g_1 g_2 \cdots g_{s-2} g_s g_{s-1})].
\]

The proof of this lemma is easy. We just substitute \( A \equiv g_1 g_2 \cdots g_{s-2}, \ B \equiv g_{s-1}, \) and \( C \equiv g_s \) into the Fricke identity Eq. (23).