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Letter

Optical property of a quasi-periodic multilayer

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Abstract

An optical property of a quasi-periodic multilayer constructed of two types of layer, A and B, is discussed. For light propagation through a quasi-periodic medium, we find that the transmission coefficient has, for the wavelength of light, many barriers where the transmission coefficient vanishes.

1. Introduction

Since Kohmoto, Kadanoff and Tang (KKT) [1] and Ostlund *et al.* [2] presented their theories of quasi-periodicity, a number of papers have studied applications [3–6]. However, it was the discovery of quasi-crystals by Schechtman *et al.* [7] in 1984 that gave a dramatic experimental demonstration of quasi-periodicity. In 1985, Merlin *et al.* [8] designed a multilayer device made by the quasi-periodic arrangement of two different types of layers of materials such as GaAs/AlGaAs.

However, the comparison between theory and experiment has never been entirely satisfactory. The systems that have been proposed by the theorists are oversimplified and in real materials we have to take into account electronic multiple-body effects. A theory such as the KKT method is a one-electron theory, so its application to real physical systems may not be appropriate. From this point of view, it is probably better to find an optical analogue to check the theory with experiment. Then we need not worry about electronic multiple-body effects in the sample. Thus Kohmoto *et al.* [6] were led to propose experiments on a quasi-periodic multilayer of optical materials, in which only a Fibonacci multilayer was studied.

In this letter we would like to generalize the above result to any quasi-periodic lattice.

2. Light propagation in a quasi-periodic multilayer

In this section we restrict ourselves to the theory of light propagation in quasi-periodic multilayers [6]. Let us consider a multilayer in which two types of layers A and B are arranged in a quasi-periodic sequence. In order to understand light propagation in these media, first consider an interface of two layers (see Fig. 1). The electric field in A is given by

$$E = E_A^{(1)} \exp[i(\boldsymbol{k}_A^{(1)}\boldsymbol{x} - \omega t)] + E_A^{(2)} \exp[i(\boldsymbol{k}_A^{(2)}\boldsymbol{x} - \omega t)] \qquad (1)$$

The electric field in B is given by the same expression except with the A subscript replaced by B. We consider a polarization which is perpendicular to the plane of the light path (transverse electric wave) [9, 10]. The appropriate boundary condition at the interface gives

$$\begin{aligned} &E_A^{(1)} + E_A^{(2)} = E_B^{(1)} + E_B^{(2)} \\ &n_A \cos\theta_A\,(E_A^{(1)} - E_A^{(2)}) = n_B \cos\theta_B\,(E_B^{(1)} - E_B^{(2)}) \end{aligned} \qquad (2)$$

where n_A and n_B are the indexes of refraction of A and B, respectively, and the angles θ_A and θ_B are shown in Fig. 2. Snell's law is $\sin\theta_A/\sin\theta_B = n_B/n_A$. It is convenient to choose the two independent variables for the electric field as

$$\begin{aligned} &E_+ = E^{(1)} + E^{(2)} \\ &E_- = (E^{(1)} - E^{(2)})/i \end{aligned} \qquad (3)$$

Then eqn. (2) gives

$$\begin{pmatrix} E_+ \\ E_- \end{pmatrix}_B = T_{BA} \begin{pmatrix} E_+ \\ E_- \end{pmatrix}_A \qquad (4)$$

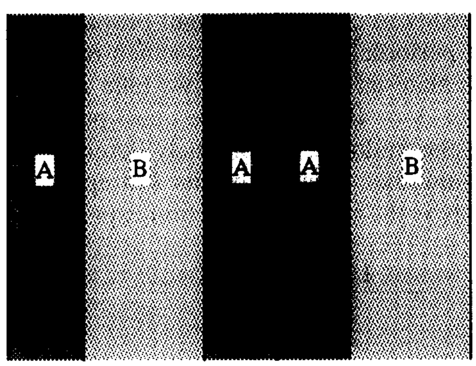

Fig. 1. A quasi-periodic multilayer. The illustration is drawn for a Fibonacci multilayer.

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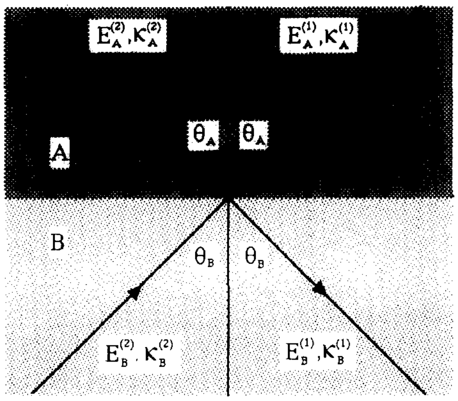

Fig. 2. The boundary condition at the interface is shown.

where the transfer matrices are given by

$$\mathbf{T}_{BA} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{n_A \cos\theta_A}{n_B \cos\theta_B} \end{pmatrix}$$

$$\mathbf{T}_{AB} = \mathbf{T}_{BA}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{n_B \cos\theta_B}{n_A \cos\theta_A} \end{pmatrix} \tag{5}$$

The matrices $\mathbf{T}_{BA}$ and $\mathbf{T}_{AB}$ represent light propagation across interfaces A→B and B→A respectively. The propagation across one layer is given by the transfer matrix

$$\mathbf{T}_A = \begin{pmatrix} \cos\delta_A & -\sin\delta_A \\ \sin\delta_A & \cos\delta_A \end{pmatrix} \tag{6}$$

for a layer of type A, and the same expression for $\mathbf{T}_B$ with δ_A replaced by δ_B. The phases δ_A and δ_B are given by

$$\delta_A = \frac{n_A k d_A}{\cos\theta_A} \qquad \delta_B = \frac{n_B k d_B}{\cos\theta_B} \tag{7}$$

where k is the wave number in vacuum, and d_A and d_B are the thicknesses of the two layers.

Now we are ready to consider the light propagation through a quasi-periodic multilayer S_k, which is sandwiched between material of type A. If we take $P_k(Q_k)$ as the number of A or B layers in the quasi-periodic multilayer, then the multilayer is uniquely determined by an irrational number λ, such that its kth approximation λ_k is given by the continued fraction expansion

$$\lambda_k \equiv \frac{P_k}{Q_k} = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ldots + 1/n_{k-1}}} = [n_0, n_1, n_2, \ldots, n_{k-1}] \tag{8}$$

Here $P_{k+1} = n_{k-1}P_k + P_{k-1}$ and $Q_{k+1} = n_{k-1}Q_k + Q_{k-1}$ with $(P_0, Q_0) = (0, 1), (P_1, Q_1) = (1, 0)$. Then we have a scaling transformation to construct this multilayer characterized by the irrational number λ, *i.e.* B→A, A→$\mathrm{BA}^{n_{k-1}}$. There are N_k layers in S_k. N_k is given recursively as $N_{k+1} = n_{k-1}N_k + N_{k-1}$ with $N_k \equiv P_k + Q_k$ and $N_1 = N_0 = 1$. In this way, if the total length of the kth generation of the multilayer coincides with the kth approximation of the irrational number, let us call the multilayer the kth primitive multilayer. The ratio N_{k+1}/N_k between the length of the $(k+1)$th multilayer and that of the kth multilayer, provides us with another continued fraction expansion $N_{k+1}/N_k = [n_{k-1}, n_{k-2}, \ldots, n_0 + 1]$. This ratio measures how fast the multilayer size grows under the scaling transformation, while λ_k measures how fast the ratio between the total numbers of A and B layers in the primitive multilayer converges.

Thus we can construct a quasi-periodic multilayer which can be represented by

$$\mathrm{L}^k(\mathrm{A}, \mathrm{B}) \equiv \mathrm{L}_{P_k, Q_k}(\mathrm{A}, \mathrm{B}) \equiv \mathrm{BA}^{a_{Q_k}}\,\mathrm{BA}^{a_{Q_{k-1}}} \ldots \mathrm{BA}^{a_1} \equiv \prod_{n=1}^{Q_k} \mathrm{BA}^{a_n} \tag{9}$$

where

$$a_n \equiv [n\lambda_k] - [(n-1)\lambda_k] \tag{10}$$

[ ] denotes the largest integer part of the number in the square bracket.

We can define two matrices for quasi-periodic multilayers:

$$\mathbf{A} \equiv \mathbf{T}_A \qquad \mathbf{B} \equiv \mathbf{T}_{AB}\mathbf{T}_B\mathbf{T}_{BA} \tag{11}$$

If we consider quasi-periodic multilayers in which the number of A layers is greater than the number of B layers, then every B layer is isolated and sandwiched by A layers so that light propagation occurs across interfaces between A and B layers. The matrices $\mathbf{T}_{AB}$ and $\mathbf{T}_{BA}$ take place at both sides of a B layer. Thus we get eqn. (11).

It can be shown that for N_k layers, *i.e.* S_k, the corresponding matrix $\mathbf{M}_k = \mathrm{L}^k(\mathbf{A}, \mathbf{B})$ is calculated as

$$\mathbf{M}_{k+1} = \mathbf{M}_k^{n_k}\mathbf{M}_{k-1} \tag{12}$$

where the initial condition is taken as $\mathbf{M}_0 = \mathbf{B}$, $\mathbf{M}_1 = \mathbf{A}$, and $\mathbf{M}_2 = \mathbf{BA}$. This equation is the same as the renormalization group equation for the quasi-periodic Schrödinger equation, which was discussed in refs. 11 and 12.

Let x_k be half the trace of $\mathbf{M}_k$. Then x_k obeys the trace map, which was discussed in refs. 11 and 12. Let (x_0, y_0, z_0) be the initial triple $\equiv (1/2\mathrm{Tr}(\mathbf{B}), 1/2\mathrm{Tr}(\mathbf{A}), 1/2\mathrm{Tr}(\mathbf{AB}))$, we can define the trace map in the following way:

$$(x_k, y_k, z_k) = \mathbf{T}_{n_{k-1}} \mathbf{T}_{n_{k-2}} \dots \mathbf{T}_{n_0}(x_0, y_0, z_0) \quad (13)$$

where we have used

$$\mathbf{T}_n(x, y, z) \equiv \{y, U_{n-1}(y)z - U_{n-2}(y)x, U_n(y)z - U_{n-1}(y)x\} \quad (14)$$

Here $U_n(y)$ is the nth Chebyshev polynomial of the second kind, which is defined by $U_{n+1}(y) = 2yU_n(y) - U_{n-1}(y)$ with $U_0(x) = 1$, $U_{-1}(x) = 0$. The scaling transformations preserve the invariant

$$I \equiv \frac{\Lambda - 2}{4} = x^2 + y^2 + z^2 - 2xyz - 1 \quad (15)$$

where $\Lambda \equiv \mathrm{Tr}(\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{B}\mathbf{A})$.

This invariant for light propagation represents the strength of the effect of quasi-periodicity. If we take the initial triple, then I is explicitly written as

$$I = \frac{1}{4}\sin^2 \delta_A \sin^2 \delta_B \left(\frac{n_A \cos\theta_A}{n_B \cos\theta_B} - \frac{n_B \cos\theta_B}{n_A \cos\theta_A}\right)^2 \quad (16)$$

For the case of $n_A = n_B$ there is no quasiperiodicity so $I = 0$ as expected. Otherwise I is always positive. Therefore, most of the physics of light propagation in quasi-periodic multilayers is very similar to that of an electron on a quasi-periodic lattice. The light wave—similar to the electron—can be localized by quasi-periodicity. We can thus call this phenomenon the quasi-localization of light [6].

Using the multiplication of the transfer matrices, we can define the transmission and reflection coefficients T_k and R_k in terms of the matrix $\mathbf{M}_k$ as

$$T_k \equiv \frac{4}{\|\mathbf{M}_k\|^2 + 2}, \qquad R_k \equiv \frac{\|\mathbf{M}_k\|^2 - 2}{\|\mathbf{M}_k\|^2 + 2} \quad (17)$$

where we have defined

$$\|\mathbf{M}_k\|^2 \equiv \mathrm{Tr}({}^t\mathbf{M}_k \mathbf{M}_k) \quad (18)$$

Here Tr denotes the trace of the matrix, ${}^t\mathbf{M}_k$ is the transpose of $\mathbf{M}_k$, and the particular combination $\mathrm{Tr}({}^t\mathbf{M}_k\mathbf{M}_k)$ is equal to the sum of the squares of the four matrix elements of $\mathbf{M}_k$. This is a quantity measured experimentally and has an important structure with respect to a variation of either the wavelength of the light or the number of layers.

3. Transmission and reflection coefficients

Let us consider the simplest experimental setting. Take the incidental light to be normal (*i.e.* $\theta_A = \theta_B = 0$) and choose the thickness of the layers to give $\delta_A = \delta_B = \delta$ (*i.e.* $n_A d_A = n_B d_B$). In this setting one can have the invariant I as

$$I = \frac{1}{4}\sin^4 \delta \left(\frac{n_A}{n_B} - \frac{n_B}{n_A}\right)^2 \quad (19)$$

where we have used the initial triple

$$(x_0, y_0, z_0) \equiv (\tfrac{1}{2}\mathrm{Tr}(\mathbf{B}), \tfrac{1}{2}\mathrm{Tr}(\mathbf{A}), \tfrac{1}{2}\mathrm{Tr}(\mathbf{AB})) = \left(\cos\delta, \cos\delta, \cos^2\delta - \frac{1}{2}\left(\frac{n_A}{n_B} + \frac{n_B}{n_A}\right)\sin^2\delta\right)$$

For $\delta = m\pi$ ($\frac{1}{2}$ wavelength layer) we have $I = 0$ and the transmission is perfect. For $\delta = (m + \frac{1}{2})\pi$ ($\frac{1}{4}$ wavelength layer), I is a maximum and the quasi-periodicity is most effective.

We can now state a simple approach to determining the nature of the transmission coefficient for any quasi-periodic multilayer. When we fix the value of k, we can regard the multilayer as being periodic with period N_k. In other words, if we assume that the kth primitive multilayer in which there are N_k layers is repeated to make an infinite multilayer, we can regard N_k layers as the unit multilayer for the infinite multilayer. Then we can impose the Bloch theorem for the electric field in the multilayer, *i.e.*

$$E_{n+N_k}{}^{(i)} = \exp\{iqN_k\} E_n{}^{(i)} \qquad i = 1, 2$$

Thus, according to the boundary condition for the electric field in eqn. (2), we can express the trace of the matrix $\mathbf{M}_k$ as

$$x_k = \cos(N_k q) \quad (20)$$

where the Brillouin zone for the light is given by $-\pi/N_k \leqslant q \leqslant \pi/N_k$. If the x_k value determined by the trace map lies in the interval $[-1, 1]$, then the corresponding wavenumber of the light is allowed. If $|x_k| > 1$, then the wavenumber is forbidden.

Let us look back at the expression for the transmission coefficient T_k (eqns. (17) and (18)). The sum of the squares of the four elements of the transfer matrix $\mathbf{M}_k$ is related to the trace of the matrix by

$$\|\mathbf{M}_k\|^2 = \mathrm{Tr}({}^t\mathbf{M}_k\mathbf{M}_k) \equiv a^2 + b^2 + c^2 + d^2 = [\mathrm{Tr}(\mathbf{M}_k)]^2 + (b - c)^2 - 2 \quad (21)$$

where we have used $\det \mathbf{M}_k \equiv ad - bc = 1$. The sum of the squares of the four elements is in the denominator of the transmission coefficient. If the trace diverges under the scaling transformations, then the transmission coefficient vanishes and conversely the reflection coefficient becomes unity.

The nature of the dynamical map on the invariant surface is as follows. The trace diverges at almost all points on the surface under the scaling transformations [11]. The initial triple escapes to one of the four horns where, if I is positive, there are four horns of the invariant surface. In order to indicate which horn the point diverges to, we can take four colours. Thus, the escaping points on the surface can be painted with four colours, while the non-escaping points remain as the boundaries of the coloured regions (see Fig. 3).

Therefore, at almost all wavenumbers, or wavelengths, the transmission coefficient vanishes where the incidental light is perfectly reflected so that the system becomes a barrier for light propagation. This situation gives infinitely many barriers in wavelength, so that the curve of the transmission coefficient with respect to wavelength becomes singular continuous. Because we know that gaps in the energy of an electron can be labelled with four colours [11], the barriers in wavelength of the light can be labelled with four colours as well.

We can see that there are many dips in the transmission coefficient curves. The higher the generation of the multilayer is, the more abundant and deeper are the dips, each position of which corresponds to that of the coloured regions on the invariant surface. The dips in the transmission coefficient deepen towards zero to make barriers in the wavelength as the generation of the multilayer becomes increasingly higher. The scaling nature of the transmission coefficient is clear around 1.5π, where quasi-periodicity is most effective. This feature comes from the scaling nature of the trace map on the invariant surface.

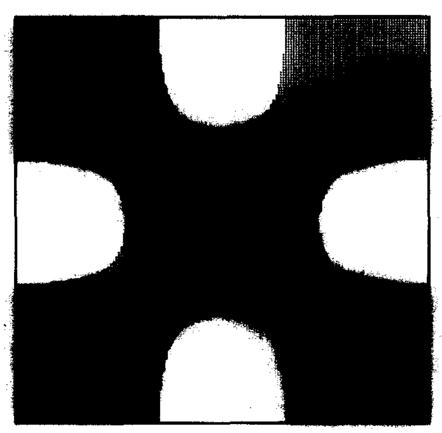

Fig. 3. The coloured invariant surface is shown for a Fibonacci case with $I = 1$.

Finally, in order to show the above gap labelling, the locations of the barriers in the wavelength for the transmission coefficient are shown in Fig. 4 for $n_A n_B = 1.5$ and in Fig. 5 for $n_A/n_B = 2.0$. We can see a very similar location of the barriers in the transmission coefficients to that of the band gaps in the electronic bands [11].

In conclusion, we have presented a theory of light propagation in a quasi-periodic multilayer. The transmission coefficient is singular continuous and has many barriers in almost all wavelengths where light is

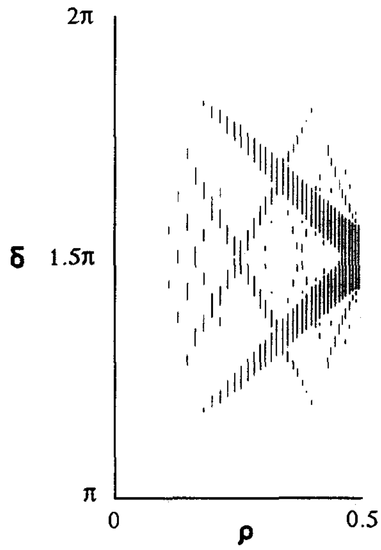

Fig. 4. The location of the barriers in the wavelength for the transmission coefficients for $n_A/n_B = 1.5$. The vertical line indicates δ between π and 2π, while the horizontal line indicates the density of B to A layers.

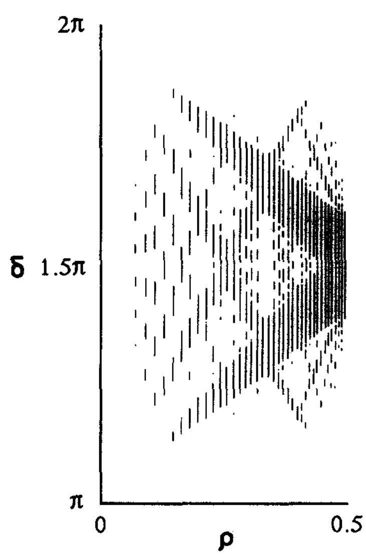

Fig. 5. The location of the barriers in the wavelength for the transmission coefficients for $n_A/n_B = 2.0$. The vertical line indicates δ between π and 2π, while the horizontal line indicates the density of B to A layers.

perfectly reflected. This feature enables us to investigate the theory experimentally.

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Appendix A

Here we give a simple program by which the readers are able to generate and calculate the barriers in wavelength.

(1) Define the initial triple (x_0, y_0, z_0), fixing the parameters δ, n_A and n_B.

(2) Define the density of B in an RQP lattice ρ; ρ is related to λ by $\rho = 1/(1+\lambda)$.

(3) Make a continued fraction of λ such that $\lambda = [n_0, n_1, \ldots, n_{k-1}]$.

(4) Generate a string R such that $R = XL^{n_{k-1}}XL^{n_{k-2}} \ldots XL^{n_1}XL^{n_0}$, where $L^n = L \ldots L$.

(5) Read the above string from the right to the left, and each time we read L, go to subroutine MAPL; otherwise, go to subroutine MAPX.

(6) Make the judgement: If $|x| \leqslant 1$, then δ is allowed, otherwise δ is forbidden.

(7) Plot δ with respect to ρ.

Subroutine MAPL:

$(x_1, y_1, z_1) = (z_0, y_0, 2y_0z_0 - x_0)$

$(x_0, y_0, z_0) = (x_1, y_1, z_1)$: Return

Subroutine MAPX:

$(x_1, y_1, z_1) = (y_0, x_0, z_0)$

$(x_0, y_0, z_0) = (x_1, y_1, z_1)$: Return