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Haldane liquid with mutual exclusion statistics

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I study basic properties of a Haldane liquid with mutual exclusion statistics (MES), which is a liquid of quasiparticles interacting only through the MES that are assumed to be parameters as usually done in the definition of Haldane statistics. By this approximation for the statistics, I obtain the following: The ground state of the system has multiple pseudo–Fermi surfaces for all species of quasiparticles when $T=0$, where the pseudo–Fermi surface for each species exhibits the particle-hole asymmetry. I define the dressed energies so as to satisfy the Sutherland-Wu (SW) functional equations that are the defining relations for the MES, from which the elementary excitations near the ground state are given. And finally, I show that the SW equations have an unusual type of duality such as spin-charge duality in the system.

I. INTRODUCTION

Fractional exclusion statistics[1–3] (FES) have been intensively studied for recent years and played a very important role to understand strongly interacting many-body systems such as the Tomonaga-Luttinger model (TLM),[4,5] the Calogero-Sutherland model (CSM),[6] the Haldane-Shastry model (HSM),[7] and the T-J model[8–11] (TJM) in one dimension, the fractional quantum Hall effect (FQHE) in two dimensions,[12–14] and the high-T_c superconductivity (HTSC) in three dimensions.[15]

In the CSM and the HSM,[6,7] the system of interacting particles with $1/r^2$ long-ranged two-body interaction behaves like an ideal gas of quasiparticles with FES. In the former, the coupling constant in the interaction between the particles defines the FES of the quasiparticles while in the latter the system is a semion gas. This situation is also realized in the TLM.[4,5] In the TJM the system is described as a mixture of gases of spinons and holons with semion statistics.[8–11] In the FQHE,[12,13] the system of interacting particles with the statistical gauge field under a very strong uniform magnetic field behaves like a gas or liquid of quasiparticles with FES. In the odd (even) denominator filling systems the interacting electrons behaves like a boson (fermion) gas of quasiparticles with some fictitious flux.[13,14] In the HTSC, as was emphasized by Anderson,[15] the concept of spin-charge separation is crucially important in these systems. It has been believed to be very significant and responsible for the origins of the HTSC. Spin-charge separation is essentially realized by two types of excitations – spin and charge excitations – in the system. So, the system behaves like a mixture of the two types of quasiparticles of spin and charge.

A microscopic derivation for the spin-charge separation was first carried out on the TLM,[4] the CSM,[6] the HSM,[7] and the TJM.[8–11] The spin-charge separation was stemmed out as a consequence of the concept of Luttinger liquids in one dimension.[5] Since spin and charge excitations are not independent of each other, this effect can be taken care of by mutual FES (MFES) between the two excitations. This situation was expected to be correct even for higher-dimensional systems.[16,17] However, the system having spin-charge separation is not only the system but there are many other systems where MFES plays an important role. Therefore, it belongs to more broad systems of multispecies quasiparticles with MFES.

Such multicomponent systems have been studied in the various low-dimensional systems for a long time.[18–27] Sutherland[18] first considered the one-dimensional multicomponent systems of a mixture of fermions and bosons by using the generalized Bethe ansatz method (BAM).[19] And Schlottmann used it to obtain the exact result for the one-dimensional TJM.[8] Also, the multicomponent systems in the FQHE have been studied by many authors.[20–27] Thus, this class of systems seems very important in the study of the strongly correlated systems.

To understand the physics of such multicomponent systems, it is necessary to describe the system as a mixture of quantum gases or liquids with MFES, since this approach enables one to apply the quantum statistical mechanics (QSM) to the problem. Recently, the concept of quantum liquids with the pure FES was reformulated by Sutherland[28] by using an electrostatic analogy to the FES, and by Iguchi[29] and Isakov[30] in terms of the language of Landau's Fermi liquid theory.[31,32] There the liquid of quasiparticles with FES in the sense of Haldane and Wu[1,2] was called a Haldane liquid.[29] As the Fermi liquid theory introduced the famous concept of quasiparticles with Fermi-Dirac statistics (FDS) in a Fermi liquid in order to describe electrons in a metal, the Haldane liquid theory naturally introduces the concept of quasiparticles with Haldane-Wu statistics (HWS) in order to describe quasiparticles in a liquid with HWS. Here, I would like to emphasize that I discriminate by the word ''HWS'' from the more general definition of FES, so that the HWS describes a FES only with constant parameters, not with functions.

In spite of the successful applications of such a concept to a Haldane liquid with a single HWS of statistical parameter g (i.e., the g-on liquid), its application to a liquid of multispecies quasiparticles with mutual HWS (MHWS) has not been so well-known yet. To understand thermal properties of such a system of multispecies quasiparticles with MHWS, it is inevitably necessary to consider QSM of the system in order to obtain the equation of state for the system. This is the high temperature limit of the system. However, it has

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been extremely difficult to do so except for pure HWS cases,[3] since MFES is given by a set of functional equations for the Wu's distribution functions.[2] On the other hand, to understand the basic physical properties of the Haldane liquid with MHWS, it is necessary to consider the ground state of the system at $T=0$. This is the low-temperature limit of the system. This has also been not so well known except for the pure HWS case and so are the excitations as well, since the momentum distribution function for each species of quasiparticles is a very complicated function through the Wu's functional equations.[2] In this way, the system of a liquid of multispecies quasiparticles seems very important in the study of the strongly correlated systems as well. In this paper, I will study the basic properties of a Haldane liquid with MHWS – a mixture of multispecies quasiparticles interacting via MHWS.

The organization of the present paper is the following: In Sec. II, I will first summarize the main results of the paper. In Sec. III, I will introduce the formalism for a Haldane liquid. In Sec. IV, I will discuss the ground state. In Sec. V, I will obtain the dressed energies from the Wu's functional equations. In Sec. VI, I will prove the generalized Luttinger type theorem. In Sec. VII, I will consider the excitations near the ground state. In Sec. VIII, I will discuss thermodynamics of the system. In Sec. IX, I will show a type of duality in the theory. In Sec. X, I will apply our formalism to a quantum liquid of two species quasiparticles with MHWS. In Sec. XI, I will give a discussion on our theory. Finally, in Sec. XII, a conclusion is made.

II. MAIN RESULTS

To ease the reference I first summarize the basic properties of the Haldane liquid with MHWS as follows.

(1) *There is a multiple pseudo-Fermi surface structure at low temperature*: As is known, when the MHWS is absent each species of quasiparticles forms its own pseudo-Fermi surface associated to the pure HWS. If the MHWS is present, then such pseudo-Fermi surfaces are obviously affected by the mixing of the MHES. However, as the result, there appears another set of pseudo Fermi surfaces.

(2) *The particle-hole asymmetry universally exists for each pseudo-Fermi surface*: In such a pseudo-Fermi surface newly formed for each species of quasiparticles, the momentum distribution of pseudomomenta for the quasiparticles – momentum density – inside the pseudo-Fermi surface is different from that outside the pseudo-Fermi surface. This asymmetry gives rise to a *fractionally charged* excitation in the system.

(3) *The dressed energies are obtained from Wu's functional equations*[2] *at low temperature*: Following the argument of Yang-Yang[33] for the BAM, I define the dressed energies such that they satisfy the Wu's functional relations. Solving the Wu's equations around $T=0$ the dressed energies are obtained. In a Fermi liquid the quasiparticle energies are not affected by the presence of other quasiparticles. However, in a Haldane liquid with MHWS, the quasiparticle energies are strongly affected by the presence of other species of quasiparticles via statistical interaction of MHWS. These affected energies are the dressed energies. So, this situation may cause a dramatic change of the spectrum for the quasiparticles in the Haldane liquid.

(4) *The sum of all effective volumes of pseudo-Fermi spheres per statistics is conserved under the introduction of the statistical interaction between quasiparticles – the generalized Luttinger-type theorem*: One of the main issues in condensed matter physics is the validity of the Luttinger theorem for interacting many-body systems.[32] In our system of a liquid of multispecies quasiparticles with MHWS it is generalized to a more generalized version. In this case, one cannot obtain such a simple Luttinger theorem for a single Fermi surface case, apart from the case of the single HWS. Instead, I obtain a *sum rule* for the multi-pseudo-Fermi surfaces, which can be regarded as a generalization of the Luttinger theorem.

(5) *Elementary excitations near the ground state are described as an excitation from each pseudo-Fermi surface*: Since there are the multiple pseudo-Fermi surfaces at the ground state of the system, an elementary excitation near the ground state is nothing but a collection of excitations of a quasiparticle from an occupied state inside the pseudo-Fermi surface to an unoccupied state outside the pseudo Fermi surface – a pair of particle-hole.

(6) *Thermodynamics of the system is governed by the generalized cluster expansion*: In the high temperature limit of the system, the Wu's functional equations can be solved for the momentum distribution functions by using the generalized Lagrange theorem for a multi-complex variable functions and using the distribution functions the equation of state is represented in terms of language of the generalized cluster expansion. This was recently performed by Iguchi[34] and studied by Isakov and coworkers[27] using computer algebra.

(7) *The system consists of a new type of duality such as spin-charge duality*: The Wu's functional equations for the FES represented in terms of w_j are nothing but the Sutherland's equations represented in terms of ζ_j for the CSM by changing variables such as $\zeta_j = 1 + 1/w_j$.[34] These Sutherland-Wu (SW) equations have a new type of symmetry in the space of the HWS parameters. In our model system, since the naming of the species is free, such a symmetry may describe a duality of the system. If one species is spin and the other species is charge, then the symmetry describes the spin-charge duality. There is a class of such a symmetry in the system.[35,36]

III. FORMALISM

Let us consider the system of a Haldane liquid of S-species with pure and mutual FES parameters, $g_{ab}(\mathbf{p},\mathbf{p}')$ $(a,b=1,\ldots,S)$ where $\mathbf{p}$ and $\mathbf{p}'$ are two momenta in the system. The total number N, energy E and momentum $\mathbf{P}$ and the entropy S of the system of quasiparticles are given by $N=\Sigma_{a,\mathbf{p}} n_a(\mathbf{p})$, $E=\Sigma_{a,\mathbf{p}}\varepsilon_a^{(0)}(\mathbf{p}) n_a(\mathbf{p})$, $\mathbf{P}=\Sigma_{a,\mathbf{p}}\mathbf{p} n_a(\mathbf{p})$, $S=k_B\Sigma_{a,\mathbf{p}}\{[n_a(\mathbf{p})+n_a^*(\mathbf{p})]\ln[n_a(\mathbf{p})+n_a^*(\mathbf{p})] - n_a(\mathbf{p})\ln n_a(\mathbf{p}) - n_a^*(\mathbf{p})\ln n_a^*(\mathbf{p})\}$, respectively. Here, $\varepsilon_a^{(0)}(\mathbf{p})$ is the one-particle spectrum of quasiparticle with species a when no MFES, $n_a(\mathbf{p})$ is the momentum distribution function of quasiparticles of species a and $n_a^*(\mathbf{p})$ the hole distribution function for species a and k_B the Boltzmann constant. The MFES is now defined through

$$n_a(\mathbf{p})+n_a^*(\mathbf{p})=1-\sum_{b,\mathbf{p}'}[g_{ab}(\mathbf{p},\mathbf{p}')-\delta_{ab}\delta_{\mathbf{p},\mathbf{p}'}]n_b(\mathbf{p}') \tag{1}$$

for $a,b=1,\ldots,S$.

Let us take the extremum of the generalized thermodynamic potential $\Omega(\equiv E-\Sigma_a\mu_a n_a(\mathbf{p})-TS)$ as $\delta\Omega=0$ where μ_a is the chemical potential for quasiparticles of species a. After some algebra using the definition of $w_a(\mathbf{p})=n_a^*(\mathbf{p})/n_a(\mathbf{p})$,[29] I obtain the equation of state, $PV(=-\Omega)=k_BT\Sigma_{a,\mathbf{p}}\ln(1+1/w_a(\mathbf{p}))$ and $N=\Sigma_{a,\mathbf{p}}n_a(\mathbf{p})=\Sigma_{a,\mathbf{p}}z_a\partial/\partial z_a(P/k_BT)$ in terms of $w_a(\mathbf{p})$, which satisfy the *Wu's functional equations*[2] for the Haldane liquid as

$$(1+w_a(\mathbf{p}))\prod_{b,\mathbf{p}'}\left(\frac{w_b(\mathbf{p}')}{1+w_b(\mathbf{p}')}\right)^{g_{ba}(\mathbf{p}',\mathbf{p})}=\frac{1}{x_a(\mathbf{p})} \tag{2}$$

for $a,b=1,\ldots,S$. Here $x_a(\mathbf{p})=e^{\beta(\mu_a-\varepsilon_a^{(0)}(\mathbf{p}))}=z_a e^{-\beta\varepsilon_a^{(0)}(\mathbf{p})}$, with $\beta=1/k_BT$ and z_a the fugacity for quasiparticles of species a. And I have $n_a(\mathbf{p})=\hat{D}_{S,a}(\mathbf{p})/D_S(\mathbf{p})$ where $D_S(\mathbf{p})=\det[w_a(\mathbf{p})\delta_{ab}\delta_{\mathbf{p},\mathbf{p}'}+g_{ab}(\mathbf{p},\mathbf{p}')]$ and $\hat{D}_{S,a}(\mathbf{p})$ is given by replacing all elements in the $(a,\mathbf{p})$ column by 1.

For example, when the MHWS interaction between the species of quasiparticles is taken care of by $g_{ab}(\mathbf{p},\mathbf{p}')=g_{ab}\delta_{\mathbf{p},\mathbf{p}'}$, I get $D_S(\mathbf{p})=\det[w_a(\mathbf{p})\delta_{ab}+g_{ab}]$ and $\hat{D}_{S,a}(\mathbf{p})$ is defined by replacing all components in the ath column by 1.

I would like to remark the following: Fukui and Kawakami[26] have introduced a similar approach in the one-dimensional multicomponent systems and their approach is maintained in even higher-dimensional systems.[3,29] However, the essential difference between their approach and ours is that in our approach for a Haldane liquid in higher dimensions one can treat not only a pseudo-Fermi surface with spherical symmetry but also the one with any geometry. On the other hand, in their one-dimensional systems the pseudo-Fermi surfaces are only *points*. Therefore, more varieties may appear in the Haldane liquids in higher dimensions. In this paper, I restrict myself to consider only the spherical pseudo-Fermi surface case.

IV. THE GROUND STATE

Let us consider the ground state of the system at $T=0$. In this limit, $x_a(\mathbf{p})=e^{\beta(\varepsilon_a^{(0)}(\mathbf{p})-\mu_a)}=0(\infty)$ if $\varepsilon_a^{(0)}(\mathbf{p})-\mu_a\leqslant 0$ (>0) where $\mu_a=\varepsilon_{F,a}$, the pseudo Fermi energy for species a. This imposes a condition for $w_a(\mathbf{p})$ by Eq. (2) such that $w_a(\mathbf{p})=0(\infty)$ if $\varepsilon_a^{(0)}(\mathbf{p})-\mu_a\leqslant 0(>0)$. Using this to $n_a(\mathbf{p})$ at the $T=0$ temperature, I find

$$n_a(\mathbf{p})=\frac{|\hat{G}_{S,a}|}{|G_S|}\equiv\frac{1}{\tilde{g}_{aa}}(=0) \tag{3}$$

for all $\varepsilon_a^{(0)}(\mathbf{p})-\mu_a\leqslant 0(>0)$ $(a=1,\ldots,S)$, where I have defined $G_S=(g_{ab})$, $|G_S|=\det(g_{ab})=D_S|_{w_a=0}$ and $|\hat{G}_{S,a}|=\hat{D}_{S,a}|_{w_a=0}$ such that $\mathrm{sign}(|G_S|)=\mathrm{sign}(|\hat{G}_{S,a}|)$ since the density $n_a(\mathbf{p})$ must be positive and $\tilde{g}_{aa}$ represents the effective statistics for quasiparticles of species a. The above result means that *there is the particle-hole asymmetry in each pseudo-Fermi sphere in the ground state at* $T=0$, which is the generalization of the result for a g-on case: $n(\mathbf{p})=1/g(=0)$ if $\varepsilon\leqslant(>)\mu=\varepsilon_F$. Furthermore, when all $\varepsilon_a^{(0)}(\mathbf{p})-\mu_a\leqslant 0$, I find

$$n(\mathbf{p})\equiv\sum_a n_a(\mathbf{p})=\sum_{a,b}(G_S^{-1})_{ab}=\mathbf{d}_S^t G_S^{-1}\mathbf{d}_S, \tag{4}$$

where $\mathbf{d}_S$ is an S dimensional column vector defined by $\mathbf{d}_S=(1,\ldots,1)^t$ and $\mathbf{d}_S^t$ its transposed vector.

This result exactly coincides with that of the Halperin's ground state $\Psi_{m_+,m_-,n}$ for the multi-quasiparticle systems in the FQHE (Ref. 20) if I identify as $m_+=g_{11},m_-=g_{22},n=g_{12}=g_{21}$. The above generalized results of Eqs. (3) and (4) to the multi-species case coincide with those for the TL liquid using the BAM,[21] the edge states in the FQH liquid,[22] and the FQH liquid using the hierarchical method[14,23,24] and the conformal field theory.[25]

Substituting the above into the expressions for the total number and the total energy, I find

$$N_a=V\int\frac{d^Dp}{(2\pi\hbar)^D}n_a(\mathbf{p})=\frac{VV_{F,a}}{\tilde{g}_{aa}(2\pi\hbar)^D}, \tag{5}$$

where $V_{F,a}$ is the volume of the pseudo-Fermi sphere for species a given by $V_{F,a}=\int_{|\mathbf{p}|\leqslant p_{F,a}}d^Dp=\pi^{D/2}/\Gamma(D/2+1)p_{F,a}^D$ with $\Gamma(s)$ the Γ function, and V and D the volume and the dimension of the system, respectively. Solving Eq. (5) for $p_{F,a}$, I obtain the pseudo-Fermi momentum $p_{F,a}=[\tilde{g}_{aa}(2\pi\hbar)^D/\pi^{D/2}\Gamma(D/2+1)d_a]^{1/D}$ where $d_a=N_a/V$. Hence, the pseudo-Fermi energy $\varepsilon_{F,a}$ is given by $\varepsilon_{F,a}=\hbar^2p_{F,a}^2/2m_a$ for the parabolic band. Similarly, I get the ground state energy for quasiparticle of species a

$$E_a=\frac{V}{\tilde{g}_{aa}}\int_{|p|\leqslant p_{F,a}}\frac{d^Dp}{(2\pi\hbar)^D}\varepsilon_a^{(0)}(\mathbf{p}). \tag{6}$$

This obviously depends upon the explicit form of the one-particle spectrum $\varepsilon_a^{(0)}(\mathbf{p})$ for a free quasiparticle without MHWS interaction, $g_{ab}(a\neq b)$. The total momentum of the system $\mathbf{P}=0$ since the ground state momentum distribution is symmetric for $|\mathbf{p}|\leqslant p_{F,a}$.

V. DRESSED ENERGY

Another representation is applicable to describe the above system. For this purpose, I assume that all quasiparticles of species a are *fermions* with the *dressed energy* $\varepsilon_a(\mathbf{p})$. Suppose that the momentum distribution function $n_F(\mathbf{p})$ is given as the FD distribution function: $n_F(\mathbf{p})=1/(e^{\beta\varepsilon_a(\mathbf{p})}+1)$, which is obtained from defining $w_F(\mathbf{p})=n_F^*(\mathbf{p})/n_F(\mathbf{p})=e^{\beta\varepsilon_a(\mathbf{p})}$ and $n_F(\mathbf{p})+n_F^*(\mathbf{p})=1$. Substituting these into Eq. (2), I find for the dressed energy

$$\varepsilon_a(\mathbf{p})=\varepsilon_a^{(0)}(\mathbf{p})-\mu_a+k_BT\sum_b(g_{ba}-\delta_{ab})\ln(1+e^{-\beta\varepsilon_b(\mathbf{p})}). \tag{7}$$

This is called the thermal BAM (TBAM) and was first introduced by Yang and Yang[33] for the exactly solvable many-body systems a long time ago.[37]

The second term in the right hand side of Eq. (7) essentially describes the contribution to the energy at momentum $\mathbf{p}$ coming from other quasiparticles with MHWS as the sum of the effective pressures, $P_a(\mathbf{p})=k_BT\ln(1+e^{-\beta\varepsilon_a(\mathbf{p})})$. In the $T=0$ limit, $k_BT\ln(1+e^{-\beta\varepsilon_a(\mathbf{p})})=-\varepsilon_a(\mathbf{p})(=0)$ if $\varepsilon_a(\mathbf{p})<0(>0)$. Equation (7) together with this condition provides, in general, 2^S cases. Let us write $(+,-,\ldots,+)$ according to whether or not $\varepsilon_a(\mathbf{p})>0$ for $a=1,\ldots,S$, such as $(-,\ldots,-)$ $[(+,\ldots,+)]$ for $\varepsilon_a(\mathbf{p})\leqslant 0(>0)$ for all $a=1,\ldots,S$. Hence, from Eq. (7), for the case of $(-,\ldots,-)$ I find $\Sigma_b g_{ba}(\varepsilon_b(\mathbf{p})-\mu_b)=\varepsilon_a^{(0)}(\mathbf{p})$ for $a,b=1,\ldots,S$. Solving for the dressed energy by using Cramer formula, I get

$$\varepsilon_a(\mathbf{p})=\sum_b (G_S^{-1})_{ba}[\varepsilon_b^{(0)}(\mathbf{p})-\mu_b], \tag{8}$$

which satisfies $\Sigma_a\varepsilon_a(\mathbf{p})=\Sigma_a(1/\tilde{g}_{aa})\varepsilon_a^{(0)}(\mathbf{p})$ apart from a constant term, as expected. And for the case of $(+,\ldots,+)$ I find $\varepsilon_a(\mathbf{p})=\varepsilon_a^{(0)}(\mathbf{p})$ for $a=1,\ldots,S$. Here the off-diagonal terms in the above equation are the nontrivial contributions from the MHWS interaction, which is not treated by the standard argument in the Fermi liquid theory.[31] The above argument has been used to the system in the FQHE.[26]

For example, consider the case of $S=2$. For the case of $(-,-)$, I find $\varepsilon_1(\mathbf{p})=(g_{22}/|G_2|)[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1]-(g_{21}/|G_2|)[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2]$, $\varepsilon_2(\mathbf{p})=(-g_{12}/|G_2|)[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1]+(g_{11}/|G_2|)[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2]$. For the case of $(+,-)$ [or $(-,+)$], I find $\varepsilon_1(\mathbf{p})=(1/g_{11})[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1]$, $\varepsilon_2(\mathbf{p})=\varepsilon_2^{(0)}(\mathbf{p})-\mu_2-(g_{12}/g_{11})$ $[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1]$ {or $\varepsilon_1(\mathbf{p})=\varepsilon_1^{(0)}(\mathbf{p})-\mu_1-(g_{21}/g_{22})[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2]$, $\varepsilon_2(\mathbf{p})=(1/g_{22})[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2]$}. And for the case of $(+,+)$, I obtain $\varepsilon_1(\mathbf{p})=\varepsilon_1^{(0)}(\mathbf{p})-\mu_1$, $\varepsilon_2(\mathbf{p})=\varepsilon_2^{(0)}(\mathbf{p})-\mu_2$.

VI. GENERALIZED LUTTINGER TYPE THEOREM

Let us consider the Luttinger type theorem[32] for a Haldane liquid with MHWS. To do so, let us first consider the volume $V_{F,a}^{(0)}$ of the pseudo Fermi sphere for free quasiparticles of species a in a Haldane gas without MHWS, which is given by Eq. (5). Suppose that the total number of free quasiparticles of all species is fixed as N. At zero temperature where $\mu_a=\varepsilon_{F,a}^{(0)}$, the HW distribution function has the following property: $n_a(\mathbf{p})=1/g_{aa}(=0)$ when $\varepsilon_a^{(0)}(\mathbf{p})\leqslant\varepsilon_{F,a}^{(0)}$ $[\varepsilon_a^{(0)}(\mathbf{p})>\varepsilon_{F,a}^{(0)}]$ for all $a=1,\ldots,S$. Substituting this into $N=\Sigma_{a,\mathbf{p}}n_a(\mathbf{p})$, I obtain $N=\Sigma_a 1/g_{aa}\Sigma_{\mathbf{p}}\theta[\varepsilon_{F,a}^{(0)}-\varepsilon_a^{(0)}(\mathbf{p})]=V/(2\pi\hbar)^D\Sigma_a V_{F,a}^{(0)}/g_{aa}$, where $\theta(\varepsilon)$ means a step function and $V_{F,a}^{(0)}/g_{aa}$ the effective volume per statistics for species a.

Let us next consider the volume $V_{F,a}$ of the pseudo Fermi sphere for quasiparticles of species a with MHWS in a Haldane liquid. In this case, the total number N is similarly given by $N=\Sigma_a 1/\tilde{g}_{aa}\Sigma_{\mathbf{p}}\theta[\varepsilon_{F,a}-\varepsilon_a(\mathbf{p})]=V/(2\pi\hbar)^D\Sigma_a V_{F,a}/\tilde{g}_{aa}$, where $\varepsilon_a(\mathbf{p})$ is the dressed energy for quasiparticles of species a. Since the total number of all quasiparticles is conserved, I conclude the following theorem:

$$\text{Theorem 1:}\quad \sum_a \frac{V_{F,a}^{(0)}}{g_{aa}}=\sum_a \frac{V_{F,a}}{\tilde{g}_{aa}}. \tag{9}$$

This theorem means that *the total sum of the effective volumes for the pseudo-Fermi spheres in an ideal Haldane gas is conserved* under the introduction of the statistical interaction between different species of quasiparticles with MHWS in a Haldane liquid. The above theorem can be thought of as a generalization of the Luttinger's theorem for a Haldane liquid with pure HWS.[29]

VII. ELEMENTARY EXCITATIONS NEAR THE GROUND STATE

Let us consider elementary excitations near the ground state, which are finite numbers of particle-hole pairs obtained by taking particles of species a from states $\mathbf{p}'_{a,\alpha}$ below the pseudo-Fermi surface of species a to states $\mathbf{p}''_{a,\alpha}$ above it. The energy and momentum are given by

$$\Delta E=E-E_0=\Sigma_{a,\mathbf{p}}\varepsilon_a^{(0)}(\mathbf{p})\,\delta n_a(\mathbf{p})$$
$$=\Sigma_{a,\mathbf{p}}\varepsilon_a(\mathbf{p})\,\delta n_F(\mathbf{p})=\Sigma_{a,\alpha}[\varepsilon_a(\mathbf{p}''_{a,\alpha})-\varepsilon_a(\mathbf{p}'_{a,\alpha})]$$

and

$$\Delta\mathbf{P}=\Sigma_{a,\mathbf{p}}\mathbf{p}\,\delta n_a(\mathbf{p})=\Sigma_{a,\mathbf{p}}\mathbf{p}\,\delta n_F(\mathbf{p})=\Sigma_{a,\alpha}(\mathbf{p}''_{a,\alpha}-\mathbf{p}'_{a,\alpha}).$$

For the sake of simplicity, let us consider the two species case of $S=2$, where there are two pseudo-Fermi surfaces given by $\varepsilon_{F,a}(a=1,2)$. Consider a particle-hole excitation, assuming $\varepsilon_{F,2}<\varepsilon_{F,1}$. In this case, there are four possibilities: (a) $\varepsilon_{F,2}<\varepsilon_{h,1}<\varepsilon_{F,1}$, $\varepsilon_{p,1}>\varepsilon_{F,1}$; (b) $\varepsilon_{h,1}<\varepsilon_{F,2}$, $\varepsilon_{p,1}>\varepsilon_{F,1}$; (c) $\varepsilon_{h,2}<\varepsilon_{F,2}<\varepsilon_{p,2}$; (d) $\varepsilon_{h,2}<\varepsilon_{F,2}$, $\varepsilon_{p,2}>\varepsilon_{F,1}$, where $\varepsilon_{p,a}(\varepsilon_{h,a})$ means the energy of particle (hole) of species $a=1,2$. (a) and (b) [(c) and (d)] represent particle-hole excitations of a quasiparticle of species $1(2)$. Accordingly for the four cases, by using the dressed energy [Eq. (8)] I find

(a) $\Delta E_1=\varepsilon_1^{(0)}(\mathbf{p}_{p,1})-\mu_1-(1/g_{11})[\varepsilon_1^{(0)}(\mathbf{p}_{h,1})-\mu_1]$;
(b) $\Delta E_1=\varepsilon_1^{(0)}(\mathbf{p}_{p,1})-\mu_1-\{(g_{22}/|G_2|)[\varepsilon_1^{(0)}(\mathbf{p}_{h,1})-\mu_1]-(g_{12}/|G_2|)[\varepsilon_2^{(0)}(\mathbf{p}_{h,1})-\mu_2]\}$;
(c) $\Delta E_2=(1/g_{11})[\varepsilon_2^{(0)}(\mathbf{p}_{p,2})-\mu_2]-\{(g_{11}/|G_2|)[\varepsilon_1^{(0)}(\mathbf{p}_{h,2})-\mu_1]-(g_{21}/|G_2|)[\varepsilon_2^{(0)}(\mathbf{p}_{h,2})-\mu_2]\}$;
(d) $\Delta E_2=\varepsilon_2^{(0)}(\mathbf{p}_{p,2})-\mu_2-\{(g_{11}/|G_2|)[\varepsilon_1^{(0)}(\mathbf{p}_{h,2})-\mu_1]-(g_{21}/|G_2|)[\varepsilon_2^{(0)}(\mathbf{p}_{h,2})-\mu_2]\}$.

On the other hand, the momentum change is obtained as

(a) $\Delta\mathbf{p}_1=\mathbf{p}_{p,1}-p_{F,1}\hat{\mathbf{p}}_{p,1}-(1/g_{11})(\mathbf{p}_{h,1}-p_{F,1}\hat{\mathbf{p}}_{h,1})$;
(b) $\Delta\mathbf{p}_1=\mathbf{p}_{p,1}-p_{F,1}\hat{\mathbf{p}}_{p,1}-[(g_{22}-g_{12})/|G_2|](\mathbf{p}_{h,1}-p_{F,1}\hat{\mathbf{p}}_{h,1})$;
(c) $\Delta\mathbf{p}_2=(1/g_{11})(\mathbf{p}_{p,2}-p_{F,2}\hat{\mathbf{p}}_{p,2})-[(g_{11}-g_{21})/|G_2|](\mathbf{p}_{h,2}-p_{F,2}\hat{\mathbf{p}}_{h,2})$;
(d) $\Delta\mathbf{p}_2=\mathbf{p}_{p,2}-p_{F,2}\hat{\mathbf{p}}_{p,2}-[(g_{11}-g_{21})/|G_2|](\mathbf{p}_{h,2}-p_{F,2}\hat{\mathbf{p}}_{h,2})$.

Here $\hat{\mathbf{p}}_{p,a}(\hat{\mathbf{p}}_{h,a})$ is a unit vector along $\mathbf{p}_{p,a}(\mathbf{p}_{h,a})$. The excitation spectrum for each case is obtained by eliminating $\mathbf{p}_{p,a}$ and $\mathbf{p}_{h,a}$ from the expressions, which provides a dispersion relation: $E=E(\mathbf{p})$ identifying as $\Delta E=E$ and $\Delta\mathbf{p}=\mathbf{p}$.

VIII. THERMODYNAMICS

Let us consider thermodynamics of the system. In QSM,[38] it is well known that the grand partition function Q for an S-species gas can be written as

$$Q=\sum_{N=0}^{\infty}\sum_{\Sigma_{a=1}^{s}N_a=N}Q_{N_1N_2\cdots N_S}z_1^{N_1}z_2^{N_2}\cdots z_S^{N_S}. \tag{10}$$

Here $Q_{N_1N_2\cdots N_S}$ $(=Q_N)$ is the microcanonical partition function of N particles and $z_a=e^{\beta\mu_a}$, μ_a the chemical potential of species a and N_a the number of species a, respectively. Thermodynamic potential Ω (the total pressure P) and the total number N of the system are given from Q as $\Omega=-PV=-k_BT\ln Q$ and $N=\Sigma_aN_a=\Sigma_az_a\partial/\partial z_a\ln Q$. Since Q is expanded as $Q=1+Q_{10\cdots 0}z_1+\cdots$, $\ln Q$ must be expanded as $(1/V)\ln Q=b_{10\cdots 0}z_1+\cdots$. Hence, Ω and N can be expanded in the form of the cluster expansion[38] as

$$-\frac{\Omega}{Vk_BT}=\frac{P}{k_BT}=\sum_{L=1}^{\infty}\sum_{\Sigma_{a=1}^{S}l_a=L}b_{l_1l_2\cdots l_S}z_1^{l_1}z_2^{l_2}\cdots z_S^{l_S}. \tag{11}$$

$$\frac{N}{V}=\sum_{L=1}^{\infty}\sum_{\Sigma_{a=1}^{S}l_a=L}Lb_{l_1l_2\cdots l_S}z_1^{l_1}z_2^{l_2}\cdots z_S^{l_S}. \tag{12}$$

where $b_{l_1l_2\cdots l_S}$ $(=b_L)$ are called cluster coefficients.[38]

In our case of the Haldane liquid with MHWS, the Q is given by $\Omega=E-TS-\Sigma_{a=1}^{S}\mu_aN_a$. Taking an extremum in order to give a most probable contribution of free energy, the momentum distribution functions are governed by the Wu's equation of Eq. (2), which can be mapped into the *Sutherland's equation*

$$\zeta_a=1+x_a\prod_b\zeta_b^{\delta_{ab}-g_{ba}} \tag{13}$$

for $a=1,\ldots,S$, where $\zeta_a=1+1/w_a$ where the distribution functions are given by $N_a=z_a\partial/\partial z_a\ln Q=\Sigma_{\mathbf{p}}z_a\partial/\partial z_a\ln\zeta_a(\mathbf{p})$. Recently, Eq. (13) has been analytically solved for ζ_a in terms of x_a such that $\zeta_a=\zeta_a(x_1,\ldots,x_S)$ by using the generalized Lagrange theorem for multi-variable complex functions. Hence, the Q and P of the system can be calculated as the cluster expansions with respect to the fugacities z_a such as

$$b_{l_1l_2\cdots l_S}=\frac{1}{L_S^{D/2}}\sum_{a=1}^{S}c^a_{l_1l_2\cdots l_S}, \tag{14}$$

where $L_S=\tau_1l_1+\tau_2l_2+\cdots+\tau_Sl_S$ with $\tau_a=m_a/m_1$, the mass ratio, and

$$c^a_{l_1l_2\cdots l_S}=C_{S,a}\Pi_{b=1}^{S}(1/l_b!)(s_b-1)(s_b-2)\cdots[s_b-(l_b-1)]$$

where $C_{S,a}$ is a determinant obtained by eliminating the (a,a)-component from $\det[l_a(\delta_{ab}-g_{ba})-s_a]$ and $s_a=\Sigma_{b=1}^{S}l_b(\delta_{ab}-g_{ab})$.[34] These types of functions have been called the *quasihypergeometric functions*,[35,36] which are the generalization of the standard hypergeometric functions of multi-complex variables.[39]

Singularities in the functions are related to the existence of a phase transition as a generalization of the Lee-Yang theorem.[40] For the one-species case of $S=1$, one can write as $P/k_BT=F(z)=\Sigma_{l=1}^{\infty}b_lz^l$ and $N/V=z\partial/\partial zF(z)=\Sigma_{l=1}^{\infty}lb_lz^l$. Then, Lee and Yang's theorem[40] tells us that *if there exists a singularity of $F(z)$ on the positive real axis of a complex z-plane, then there is a phase transition in the system*. Therefore, the convergence of the cluster expansion is exactly related to the existence of a phase transition of the system. Thus, the explicit evaluation of the cluster coefficients is very important for the theory of phase transition. Recently some efforts have been done to understand this problem of singularities in the generalized cluster expansions.[35,36]

IX. TYPE OF DUALITY

Let us consider a type of duality in the theory of the Haldane liquid with MHWS. Let us first consider the case of single species of $S=1$. When an operation preserves the form of the SW equation such that $\zeta=1+x\zeta^{1-g}\rightarrow\zeta'=1+x'\zeta'^{1-g'}$ as $g\rightarrow g'$, $x\rightarrow x'$ and $\zeta\rightarrow\zeta'$, it is called a duality. There are two kinds of duality defined by the replacement of the statistical parameter g: (1) σ: $g\rightarrow 1-g$, $x\rightarrow -x$ and $\zeta\rightarrow 1/\zeta$, *supersymmetric duality*; (2) τ: $g\rightarrow 1/g$, $x\rightarrow x^{-1/g}$ and $\zeta\rightarrow\zeta/(\zeta-1)$, *particle-hole duality*. I find that the following relations are satisfied: $\tau^2=\sigma^2=(\tau\sigma)^3=e$ (i.e., $\tau\sigma\tau=\sigma\tau\sigma$). Hence, these generators $\langle\tau,\sigma\rangle$ form a dihedral group of order six and can be regarded as *quasimodular* transformations acting on the SW equation, $\zeta=1+x\zeta^{1-g}$. These quasimodular transformations of duality also act on the cluster expansion to induce a particular symmetry for the quasihypergeometric function.[36]

For the case of $S=2$, there is a similar group structure. Denote 2×2 matrices G_2 and G'_2 by $G_2=(g_{ab})$ and $G'_2=(\delta_{ab}-g_{ab})\equiv(g'_{ab})$ such that $|G_2|=\det(g_{ab})$ and $|G'_2|=\det(g'_{ab})$ for $a,b=1,2$. Let us define the following generators:

$$eG'_2=\begin{pmatrix}g'_{11} & g'_{12}\\ g'_{21} & g'_{22}\end{pmatrix},\quad \rho_1G'_2=\begin{pmatrix}g'_{22} & g'_{21}\\ g'_{12} & g'_{11}\end{pmatrix}, \tag{15}$$

$$\sigma_1G'_2=\begin{pmatrix}1-g'_{11} & -g'_{12}\\ g'_{21} & g'_{22}\end{pmatrix},$$

$$\sigma_2G'_2=\begin{pmatrix}g'_{11} & g'_{12}\\ -g'_{21} & 1-g'_{22}\end{pmatrix} \tag{16}$$

$$\tau_1G'_2=\begin{pmatrix}\dfrac{1}{g'_{11}} & \dfrac{g'_{12}}{g'_{11}}\\ -\dfrac{g'_{21}}{g'_{11}} & \dfrac{|G'_2|}{g'_{11}}\end{pmatrix},$$

$$\tau_2G'_2=\begin{pmatrix}\dfrac{|G'_2|}{g'_{22}} & -\dfrac{g'_{12}}{g'_{22}}\\ \dfrac{g'_{21}}{g'_{22}} & \dfrac{1}{g'_{22}}\end{pmatrix} \tag{17}$$

These quasimodular generators $\langle\rho_1,\sigma_1,\sigma_2,\tau_1,\tau_2\rangle$ act on the SW equation of Eq. (13) and form the group of order 72 $(=6^2\times 2!)$ with the defining relations $\rho_1^2=\sigma_1^2=\sigma_2^2=\tau_1^2=\tau_2^2=e$, $\sigma_1\sigma_2=\sigma_2\sigma_1$, $\tau_1\tau_2=\tau_2\tau_1$, $\tau_1\sigma_1\tau_1=\sigma_1\tau_1\sigma_1$, and $\tau_2\sigma_2\tau_2=\sigma_2\tau_2\sigma_2$.[35] Thus, since one can identify species 1(2) as charge (spin) degree, this type of generators constructs a new kind of duality such as the *spin-charge duality*. The above argument can be straightforwardly generalized to the S species case. The number of such generators is given by $6^S S!$ such that they form a finite group $\mathcal{G}\cong\mathcal{S}_3^S\times\mathcal{S}_S$, which reveals a new symmetry in the theory.

X. APPLICATION TO A HALDANE LIQUID OF TWO SPECIES QUASIPARTICLES

To show the advantage of the above rather formal approach, I apply it to a two component system such as a mixture of two quantum liquids with HWS. Suppose that the momentum distribution functions of the two species are given by $n_1(\mathbf{p})$ and $n_2(\mathbf{p})$, respectively. And the statistical parameters are given by the matrix $G_2=(g_{ab})$ for $a,b=1,2$ and $|G_2|=\det(G_2)=g_{11}g_{22}-g_{12}g_{21}$. So, in this description, one can regard the species 1 (2) as ''up (down) spin.'' Although only the two species case with MHWS is concerned in this section, one can obviously generalize the theory to the multispecies case and to the more general FES with momentum dependence.

Now, thermodynamic potential $\Omega=E-\mu_1N_1-\mu_1N_1-TS$, $E=\Sigma_{a=1,2}\varepsilon_a^{(0)}(\mathbf{p})n_a(\mathbf{p})$, $N_a=\Sigma_{\mathbf{p}}n_a(\mathbf{p})$.

$$S=k_B\Sigma_{\mathbf{p}}\{[n_a(\mathbf{p})+n_a^*(\mathbf{p})]\ln[n_a(\mathbf{p})+n_a^*(\mathbf{p})]-n_a(\mathbf{p})\ln n_a(\mathbf{p})-n_a^*(\mathbf{p})\ln n_a^*(\mathbf{p})\}.$$

The HWS is introduced by the definition:

$$n_1(\mathbf{p})+n_1^*(\mathbf{p})=1-(g_{11}-1)n_1(\mathbf{p})-g_{12}n_2(\mathbf{p}), \quad (18)$$

$$n_2(\mathbf{p})+n_2^*(\mathbf{p})=1-g_{21}n_1(\mathbf{p})-(g_{22}-1)n_2(\mathbf{p}). \quad (19)$$

By taking the extremum condition $\delta\Omega=0$, I find the Wu's functional equations for the two species case:

$$w_1^{g_{11}}(1+w_1)^{1-g_{11}}\left(\frac{w_2}{1+w_2}\right)^{g_{21}}=e^{\beta(\varepsilon_1^{(0)}-\mu_1)}, \quad (20)$$

$$w_2^{g_{22}}(1+w_2)^{1-g_{22}}\left(\frac{w_1}{1+w_1}\right)^{g_{12}}=e^{\beta(\varepsilon_2^{(0)}-\mu_2)}, \quad (21)$$

where $\varepsilon_1^{(0)}=\varepsilon_1^{(0)}(\mathbf{p})$ and $w_a=w_a(\mathbf{p})$ that are defined by

$$(w_1(\mathbf{p})+g_{11})n_1(\mathbf{p})+g_{21}n_2(\mathbf{p})=1, \quad (22)$$

$$g_{12}n_1(\mathbf{p})+(w_2(\mathbf{p})+g_{22})n_2(\mathbf{p})=1. \quad (23)$$

If I use change of variables, $\zeta_a=1+1/w_a$, to the above equations, then they are transformed to the Sutherland's form:

$$\zeta_1=1+z_1\zeta_1^{1-g_{11}}\zeta_2^{-g_{21}}, \quad (24)$$

$$\zeta_2=1+z_2\zeta_1^{-g_{12}}\zeta_2^{1-g_{22}}, \quad (25)$$

where $z_a=e^{\beta\mu_a}$ are the fugacities. The momentum distribution functions are obtained by solving Eqs. (22) and (23):

$$n_1(\mathbf{p})=\frac{w_2(\mathbf{p})+g_{22}-g_{12}}{[w_1(\mathbf{p})+g_{11}][w_2(\mathbf{p})+g_{22}]-g_{12}g_{21}}, \quad (26)$$

$$n_2(\mathbf{p})=\frac{w_1(\mathbf{p})+g_{11}-g_{21}}{[w_1(\mathbf{p})+g_{11}][w_2(\mathbf{p})+g_{22}]-g_{12}g_{21}}. \quad (27)$$

Let us first consider the low temperature limit. Equations (26) and (27) provide the momentum density in the ground state when $T=0$ as

$$n_1(\mathbf{p})=\frac{g_{22}-g_{12}}{|G_2|}=\frac{1}{\tilde{g}_{11}}, \quad (\text{for } |\mathbf{p}|\leqslant p_{F,1}), \quad (28)$$

$$n_2(\mathbf{p})=\frac{g_{11}-g_{21}}{|G_2|}=\frac{1}{\tilde{g}_{22}}, \quad (\text{for } |\mathbf{p}|\leqslant p_{F,2}), \quad (29)$$

where the pseudo-Fermi momenta $p_{F,a}$ are given by $N_a=(1/\tilde{g}_{aa})V/(2\pi\hbar)^D\int_{p\leqslant p_{F,a}}d^Dp$. And the ground-state energy E_0 and the total momentum $\mathbf{P}_0$ are given by $E_0=\Sigma_{a=1,2}(1/\tilde{g}_{aa})\,V/(2\pi\hbar)^D\int_{p\leqslant p_{F,a}}\varepsilon_a^{(0)}(\mathbf{p})d^Dp$, $\mathbf{P}_0=\Sigma_{a=1,2}(1/\tilde{g}_{aa})V/(2\pi\hbar)^D\int_{p\leqslant p_{F,a}}\mathbf{p}d^Dp=0$.

Let us consider the dressed energies, following the argument in Sec. V. Let us define the dressed energies $\varepsilon_a(\mathbf{p})$ such that $w_a(\mathbf{p})=e^{\beta\varepsilon_a(\mathbf{p})}$. Substituting to Eqs. (20) and (21), I obtain

$$\varepsilon_1(\mathbf{p})-(g_{11}-1)k_BT\ln(1+e^{-\beta\varepsilon_1(\mathbf{p})})-g_{21}k_BT\ln(1+e^{-\beta\varepsilon_2(\mathbf{p})})=\varepsilon_1^{(0)}(\mathbf{p})-\mu_1, \quad (30)$$

$$\varepsilon_2(\mathbf{p})-(g_{22}-1)k_BT\ln(1+e^{-\beta\varepsilon_2(\mathbf{p})})-g_{12}k_BT\ln(1+e^{-\beta\varepsilon_1(\mathbf{p})})=\varepsilon_2^{(0)}(\mathbf{p})-\mu_2. \quad (31)$$

Consider the $T=0$ limit. For the case of $\varepsilon_1(\mathbf{p})<0$ and $\varepsilon_2(\mathbf{p})<0$, denoting by $(-,-)$, I find

$$g_{11}\varepsilon_1(\mathbf{p})+g_{21}\varepsilon_2(\mathbf{p})=\varepsilon_1^{(0)}(\mathbf{p})-\mu_1, \quad (32)$$

$$g_{12}\varepsilon_1(\mathbf{p})+g_{22}\varepsilon_2(\mathbf{p})=\varepsilon_2^{(0)}(\mathbf{p})-\mu_2. \quad (33)$$

As was obtained in Sec. V, solving for the dressed energies, I obtain

$$\varepsilon_1(\mathbf{p})=\frac{g_{22}}{|G_2|}[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1]-\frac{g_{21}}{|G_2|}[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2], \quad (34)$$

$$\varepsilon_2(\mathbf{p})=-\frac{g_{12}}{|G_2|}[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1]+\frac{g_{11}}{|G_2|}[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2]. \quad (35)$$

For the case of $(+,-)$, I find

$$\varepsilon_1(\mathbf{p})=\frac{1}{g_{11}}[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1], \quad (36)$$

$$\varepsilon_2(\mathbf{p})=\varepsilon_2^{(0)}(\mathbf{p})-\mu_2-\frac{g_{12}}{g_{11}}[\varepsilon_1^{(0)}(\mathbf{p})-\mu_1]. \quad (37)$$

Similarly for the case of $(-,+)$, I find

$$\varepsilon_1(\mathbf{p})=\varepsilon_1^{(0)}(\mathbf{p})-\mu_1-\frac{g_{21}}{g_{22}}[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2], \tag{38}$$

$$\varepsilon_2(\mathbf{p})=\frac{1}{g_{22}}[\varepsilon_2^{(0)}(\mathbf{p})-\mu_2]. \tag{39}$$

And for the case of $(+,+)$, I obtain

$$\varepsilon_1(\mathbf{p})=\varepsilon_1^{(0)}(\mathbf{p})-\mu_1, \tag{40}$$

$$\varepsilon_2(\mathbf{p})=\varepsilon_2^{(0)}(\mathbf{p})-\mu_2. \tag{41}$$

Using the above dressed energies, the excitations near the ground state have been obtained in Sec. VII. So, I omit the derivation here. The dressed energies are very important when one applies to a physical problem. For example, let us consider the TJM in one dimension. As is well known, there are the holon and spinon bands in the system.[8–11] And originally the holon spectrum is dispersionless such that the energy is a constant as $\varepsilon_h=\pi^2/6$ in the thermodynamic limit of $N\to\infty$, while the spinon spectrum is given by $\varepsilon_s=(p^2-\pi^2)/4$. So, it means that holons are originally *localized*. However, if there is statistical mixing by g_{12} and g_{21}, then holons can be *delocalized* having a dispersion of the dressed energy. Therefore, holons become able to move by the MHWS interaction between spinons and holons. This is a novel kind of *metal-insulator transition due to MHWS*.

Let us next consider thermodynamics of the system as the high-temperature limit. In this case, from Eqs. (24) and (25) one obtains the equation of state for the system in terms of the language of the generalized cluster expansions

$$\frac{P}{k_BT}=\frac{1}{\lambda^D}\sum_{l_1,l_2=0}^{\infty}{}' \; b_{l_1l_2}z_1^{l_1}z_2^{l_2}, \tag{42}$$

$$\frac{N}{V}=\frac{1}{\lambda^D}\sum_{l_1,l_2=0}^{\infty}{}' \;(l_1+l_2)b_{l_1l_2}z_1^{l_1}z_2^{l_2}, \tag{43}$$

$$\frac{M}{V}=\frac{1}{\lambda^D}\sum_{l_1,l_2=0}^{\infty}{}' \;(l_1-l_2)b_{l_1l_2}z_1^{l_1}z_2^{l_2}, \tag{44}$$

where $'$ means the $l_1=l_2=0$ is excluded in the sum and the generalized cluster coefficients $b_{l_1l_2}$ are given by

$$b_{l_1l_2}=-\frac{l_1g_{21}+l_2g_{12}}{(l_1+\tau l_2)^{D/2}}\prod_{a=1}^{2}\frac{1}{l_a!}(s_a-1)\cdots[s_a-(l_a-1)], \tag{45}$$

$$b_{l_10}=\frac{1}{(l_1+\tau l_2)^{D/2}}\frac{1}{l_1!}(s_1-1)\cdots[s_1-(l_1-1)], \tag{46}$$

$$b_{0l_2}=\frac{1}{(l_1+\tau l_2)^{D/2}}\frac{1}{l_2!}(s_2-1)\cdots[s_2-(l_2-1)], \tag{47}$$

where $s_1=l_1(1-g_{11})-l_2g_{12}, s_2=-l_1g_{21}+l_2(1-g_{22})$ and $\lambda\equiv\sqrt{2\pi\hbar^2/m_1k_BT}$, the thermal length, and $\tau=m_2/m_1$, the mass ratio (I assume $\tau=1$, below).

From the above, the first virial coefficients are given by $a_{10}=b_{10}=1$ and $a_{01}=b_{01}=1$, and the second virial coefficients a_{20}, a_{11}, and a_{02} are given by

$$a_{20}=-b_{20}=\frac{2g_{11}-1}{2^{1+D/2}}, \tag{48}$$

$$a_{11}=-b_{11}=\frac{g_{12}+g_{21}}{2^{D/2}}, \tag{49}$$

$$a_{02}=-b_{02}=\frac{2g_{22}-1}{2^{1+D/2}}. \tag{50}$$

Hence, the pressure is expanded as

$$P=nk_BT\left[1+\frac{\lambda^D}{n}(a_{20}n_1^2+a_{11}n_1n_2+a_{02}n_2^2)+\cdots\right], \tag{51}$$

where $n=n_1+n_2=(N_1+N_2)/V$. Defining $m=n_1-n_2=(N_1-N_2)/V$, I can rewrite the second term as

$$a_{20}n_1^2+a_{11}n_1n_2+a_{02}n_2^2=\tfrac{1}{4}(An^2+2Bnm+Cm^2), \tag{52}$$

where

$$A=a_{20}+a_{11}+a_{02}=\frac{1}{2^{D/2}}(g_{11}+g_{22}+g_{12}+g_{21}-1), \tag{53}$$

$$B=a_{20}-a_{02}=\frac{1}{2^{D/2}}(g_{11}-g_{22}), \tag{54}$$

$$C=a_{20}-a_{11}+a_{02}=\frac{1}{2^{D/2}}(g_{11}+g_{22}-g_{12}-g_{21}-1). \tag{55}$$

Therefore, I obtain the equation of state as

$$P=nk_BT\left[1+\frac{\lambda^D}{4n}(An^2+2Bnm+Cm^2)+\cdots\right]. \tag{56}$$

This leads to some important conclusions: (1) If $g_{11}=g_{22}$, then $B=0$. Hence, *charge degree can be separated out from spin degree*; (2) If $g_{12}=-g_{21}$, then $A=C=(1/2^{D/2})(g_{11}+g_{22}-1)$; (3) Combining the above two cases, if $g_{11}=g_{22}$ and $g_{12}=-g_{21}$, then $A=C=(1/2^{D/2})\times(2g_{11}-1)$ and $B=0$. Hence, if $g_{11}>1/2(<1/2)$, then charge and spin degrees behave like fermionic (bosonic).

XI. DISCUSSION

I would like to put some notions for our formalism of MHWS. In Eqs. (18) and (19) I have used the momentum distribution function $n_a(\mathbf{p})$ for each species. However, this can be done in an alternative way as follows. If I define the $\rho(\mathbf{p})=[n_1(\mathbf{p})+n_2(\mathbf{p})]/2$, $\rho^*(\mathbf{p})=[n_1^*(\mathbf{p})+n_2^*(\mathbf{p})]/2$, and $\sigma(\mathbf{p})=[n_1(\mathbf{p})-n_2(\mathbf{p})]/2$, then $n_1(\mathbf{p})=\rho(\mathbf{p})+\sigma(\mathbf{p})$ and $n_2(\mathbf{p})=\rho(\mathbf{p})-\sigma(\mathbf{p})$, where I have assumed that $\sigma^*(\mathbf{p})=n_1^*(\mathbf{p})-n_2^*(\mathbf{p})=0$ since the hole distribution functions $n_1^*(\mathbf{p})$ and $n_2^*(\mathbf{p})$ are supposed to be the same for the particles. Substituting these in Eqs. (18) and (19), I find

$$\rho(\mathbf{p})+\rho^*(\mathbf{p})=1-(K_{cc}-1)\rho(\mathbf{p})-K_{cs}\sigma(\mathbf{p}), \tag{57}$$

$$\sigma(\mathbf{p})=-K_{sc}\rho(\mathbf{p})-(K_{ss}-1)\sigma(\mathbf{p}), \tag{58}$$

where the parameters K_{ab} are defined by

$$K_{cc}=\frac{g_{11}+g_{22}+g_{12}+g_{21}}{2}, \tag{59}$$

$$K_{cs}=\frac{g_{11}-g_{22}-g_{12}+g_{21}}{2}, \tag{60}$$

$$K_{sc}=\frac{g_{11}-g_{22}+g_{12}-g_{21}}{2}, \tag{61}$$

$$K_{ss}=\frac{g_{11}+g_{22}-g_{12}-g_{21}}{2}. \tag{62}$$

These are very similar to the expressions in the generalized BAM for the two-component systems.[8–11,18,19] Indeed, since spin degree is not a real one, the pseudo momentum distribution of the spin degree does not contribute to the total pressure such that

$$P=\frac{k_BT}{V}\sum_{\mathbf{p}}\ln\left[1+\frac{1}{w_c(\mathbf{p})}\right], \tag{63}$$

where $w_c(\mathbf{p})=\rho^*(\mathbf{p})/\rho(\mathbf{p})=e^{\beta\varepsilon_c(\mathbf{p})}$ and $n=2\Sigma_{\mathbf{p}}\rho(\mathbf{p})$, $m=2\Sigma_{\mathbf{p}}\sigma(\mathbf{p})$, and $E=\Sigma_{\mathbf{p}}\varepsilon_c(\mathbf{p})\rho(\mathbf{p})$.

Now, the relationship between the parameters K_{ab} and the parameters A,B,C in the previous section is apparent. I obtain

$$A=\frac{2K_{cc}-1}{2^{D/2}}, \tag{64}$$

$$B=\frac{K_{cs}+K_{sc}}{2^{D/2-1}}, \tag{65}$$

$$C=\frac{2K_{ss}-1}{2^{D/2}}. \tag{66}$$

In this way, the parameters K_{ab} can be regarded as MHWS parameters for the charge and spin degrees. I now find that if K_{cc} and $K_{ss}>1/2$ $(<1/2)$, then charge and spin excitations are repulsive (attractive), respectively. And if $K_{cs}+K_{sc}=0(B=0)$, then the spin-charge separation is realized. This is a generalization of the discussion by Wu[2] for the g-on case where the second virial coefficient is given by $a_2=(1/2^{D/2})\times(2g-1)$ such that the equation of state is given by $P=nk_BT(1+\lambda^D na_2+\cdots)$.[3]

XII. CONCLUSION

In conclusion, I have discussed various aspects of the theory of a Haldane liquid with MHWS. I have shown that many properties of the Haldane liquid are shared with those of the multi-component liquid such as the TL liquid in one dimension and FQH liquid in two dimensions. I have also shown a type of duality inherent in the theory. To show the advantage of the present theory and to show what is going on in the problem, I have applied it to the Haldane liquid of two species quasiparticles in detail. And I have discussed an alternative approach using the separation of charge and spin degrees to the approach based on the two species.

The present theory seems rather mathematical since the definition of FES is certainly restricted to be the HWS where the FES are parametrized by constants and in reality there may exist more general definitions of FES as was emphasized by Anderson.[15] However, I feel that the concept of Haldane liquids with MHWS is important as a candidate in order to understand non-Fermi liquid behaviors of the strongly interacting many-body systems in higher dimensions. In this context, the relationship between the Haldane liquid presented in this paper and the bosonization approach to the non-Fermi liquids in higher dimensions[15,41,42] is worth investigating. Furthermore, it seems very important to study how the MHWS parameters are obtained from scattering matrices in the scattering process between multispecies quasiparticles in reality as was discussed in the pure HWS case.[29]

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