Singularity and Monodromy of Quasi-Hypergeometric Functions

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Dedicated to Richard Askey

1. Introduction

Let \( r, s \) be non-negative integers. Consider the series

\[
F(x) = \sum_{\nu=0}^{\infty} \frac{\Gamma(\alpha'_1 + \beta'_1 \nu) \cdots \Gamma(\alpha'_r + \beta'_r \nu)}{\Gamma(\alpha_1 + \beta_1 \nu) \cdots \Gamma(\alpha_s + \beta_s \nu) \nu!} x^\nu
\]

in \( x \in \mathbb{C} \) for suitable complex numbers \( \alpha'_1, \ldots, \alpha'_r, \alpha_1, \ldots, \alpha_s, \) and positive numbers \( \beta'_1, \ldots, \beta'_r, \beta_1, \ldots, \beta_s, \) which have the relation

\[
\beta'_1 + \cdots + \beta'_r = \beta_1 + \cdots + \beta_s + 1
\]

\( F(x) \) is convergent for \( |x| < c \) where \( c \) denotes the constant

\[
c = \beta'_1 - \beta'_r \cdots \beta'_r - \beta'_r \beta_1 \cdots \beta_s
\]

These functions including Lambert series have recently appeared in physics literatures of fractional exclusion statistics (see [8, 11, 12, 17, 18], etc.) In [7] I.M. Gelfand and M.I. Graev have studied them in a systematic way in both regular and irregular singular cases. In [2] we have formulated them only for regular singularity. In this note we restrict ourselves to this case of a single variable. It seems important to study their singularities and monodromy properties at the branch point \( x = c \), both from mathematical and physical point of views. We give an answer to this problem (see Theorem 1).

In [2] and [7] it has been shown that they are characterized by certain difference and differential equations, so that there arises a monodromy problem for solutions by an analytic continuation. We give a conjecture for this problem (see Conjecture) and give an answer in simple cases, by using integral representations for these functions.
We define two exponents $\gamma, \delta$ which will be frequently used

$$\gamma = \alpha'_1 + \cdots + \alpha'_r - \alpha_1 - \cdots - \alpha_s + s,$$

$$\delta = -\gamma + \frac{n}{2}$$

where we put $n = r + s - 1$.

For $\sigma > 0$ and $\alpha, \beta \in \mathbb{C}$, we define the Erdelyi-Kober operator $P_\sigma(\alpha, \beta)$ as

$$P_\sigma(\alpha, \beta)f(x) = \frac{1}{\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}f(t^\sigma x)dt$$

If $\Re \alpha$ or $\Re \beta$ is not positive, the RHS in (1.4) should be a finite part of divergent integral in the sense of Hadamard-Riesz.

Then as a function of $x$, $F(x)$ satisfies the following fractional differential equations (E)

$$\frac{dF(x)}{dx} = \prod_{k=1}^{r} P_{\beta'_k}(\alpha'_k + \beta'_k, -\beta_k') \prod_{k=1}^{s} P_{\beta_k}(\alpha_k, \beta_k)F(x).$$

$F(x)$ is uniquely characterized by the properties that it is holomorphic at the origin $x = 0$, satisfies (E) and the following initial condition

$$F(0) = \frac{\Gamma(\alpha'_1) \cdots \Gamma(\alpha'_r)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_s)}$$

As a function of $x$ and $\alpha'_1, \cdots \alpha'_r, \alpha_1, \cdots, \alpha_s$, $F$ also satisfies the following difference and differential equations (E*)

$$T_{\alpha'_k}F = (\alpha'_k + \beta'_k x \frac{d}{dx})F$$

$$(E^*) \quad F = (\alpha_k + \beta_k x \frac{d}{dx})T_{\alpha_k}F$$

$$\frac{dF}{dx} = T_{\beta'_1}^{e_1} \cdots T_{\alpha'_r}^{e_r} T_{\beta_1}^{e_1} \cdots T_{\alpha_s}^{e_s} F$$

where $T_{\alpha'_k}, T_{\alpha_k}, T_{\beta'_k}, T_{\beta_k}$ denote the shift operators corresponding to the displacements $\alpha'_k \rightarrow \alpha'_k + 1, \alpha_k \rightarrow \alpha_k + 1, \alpha'_k \rightarrow \alpha'_k + \beta'_k, \alpha_k \rightarrow \alpha_k + \beta_k$ respectively.

These equations have been defined in [2]. In [7] I.M. Gelfand-M.I. Graev have also studied the same equations as (E*) in a very extensive way. However they do not restrict themselves to the relation (1.2) which is essential in our note.

$F(x)$ has a branch singularity at $x = c$ and is holomorphic in $\mathbb{C} - [c, \infty)$. In the complex $x$-plane, we consider the analytic continuation of $F(x)$ along a path on the real axis from the origin to $c - \epsilon$ ($\epsilon$ a small positive number.)
As is seen from the integral representation (3.3), we can show that \( F(x) \) has a convergent power series expansion at \( x = c \),

\[
F(x) = A(x) + B(x)
\]

where

\[
A(x) = (c - x)\delta \sum_{\nu=0}^{\infty} a_{\nu}(c - x)\nu,
\]

\[
B(x) = \sum_{\nu=0}^{\infty} b_{\nu}(c - x)^{\nu}
\]

The coefficients \( a_{\nu}(\nu \geq 0) \) depending on \((\alpha', \alpha) = (\alpha'_1, \ldots, \alpha'_r, \alpha_1, \ldots, \alpha_s)\) are determined recursively as follows.

From (E*), \( a_0 = a_0(\alpha', \alpha) \) satisfies the difference equations in \( \alpha' \) and \( \alpha \)

\[
T_{\alpha'_k} a_0(\alpha', \alpha) = -\beta'_k c^\delta \cdot a_0(\alpha', \alpha) \\
T_{\alpha_s} a_0(\alpha', \alpha) = -\frac{1}{\beta_k c (\delta + 1)} \cdot a_0(\alpha', \alpha)
\]

so that \( a_0(\alpha', \alpha) \) can be expressed as

\[
a_0(\alpha', \alpha) = (\beta'_1 c)^{\alpha'_1} \cdots (\beta'_r c)^{\alpha'_r} (\beta_1 c)^{-\alpha_1} \cdots (\beta_s c)^{-\alpha_s} \Gamma(-\delta) \cdot P(\alpha', \alpha)
\]

where \( P(\alpha', \alpha) \) denotes a periodic function which is periodic with the period 1 with respect to each variable of \( \alpha'_1, \ldots, \alpha'_r, \alpha_1, \ldots, \alpha_s \).

The factor \( P(\alpha', \alpha) \) is not determined by (E*).

For general \( \nu \geq 0 \) the functions

\[
\psi_{\nu}(\alpha', \alpha) = (\beta'_1 c)^{\alpha'_1} \cdots (\beta'_r c)^{\alpha'_r} (\beta_1 c)^{-\alpha_1} \cdots (\beta_s c)^{-\alpha_s} \Gamma(-\delta - \nu)
\]

satisfy the difference equations

\[
T_{\alpha'_k} \psi_{\nu}(\alpha', \alpha) = -\beta'_k c (\delta + \nu) \psi_{\nu}(\alpha', \alpha) \\
T_{\alpha_s} \psi_{\nu}(\alpha', \alpha) = -\frac{1}{\beta_k c (\delta + \nu + 1)} \psi_{\nu}(\alpha', \alpha)
\]

\[
T_{\alpha'_1} \cdots T_{\alpha'_r} T_{\alpha_1} \cdots T_{\alpha_s} \psi_{\nu}(\alpha', \alpha) = -(\delta + \nu) \psi_{\nu}(\alpha', \alpha)
\]

If we put

\[
a_{\nu}(\alpha', \alpha) = \psi_{\nu}(\alpha', \alpha) \tilde{a}_{\nu}(\alpha', \alpha)
\]

then \( \tilde{a}_{\nu}(\alpha', \alpha) \) satisfies the recurrence relations

\[
T_{\alpha'_k} \tilde{a}_{\nu}(\alpha', \alpha) - \tilde{a}_{\nu}(\alpha', \alpha) = \frac{\alpha'_k - \beta'_k (\delta + \nu - 1)}{\beta'_k c} \tilde{a}_{\nu-1}(\alpha', \alpha)
\]

\[
T_{\alpha_k} \tilde{a}_{\nu}(\alpha', \alpha) - \tilde{a}_{\nu}(\alpha', \alpha) = \frac{\alpha_k - \beta_k (\delta + \nu)}{\beta_k c} T_{\alpha_k} \tilde{a}_{\nu-1}(\alpha', \alpha)
\]

\[
T_{\alpha'_1} \cdots T_{\alpha'_r} T_{\alpha_1} \cdots T_{\alpha_s} \tilde{a}_{\nu}(\alpha', \alpha) = \tilde{a}_{\nu}(\alpha', \alpha)
\]
These equations determine uniquely $a_0(\alpha', \alpha)$, once $a_0(\alpha', \alpha)$ is given.

Remark 1. It was pointed out by A. Gyoja that the equation (E) is a micro differential operator developed by M. Sato, T. Kawai and M. Kashiwara. As a consequence of it, one can also show that $F(x)$ has the expression (1.8). For this computation see [16].

The 1st Problem which we want to pose is to evaluate $a_0(\alpha', \alpha)$ explicitly for $F(x)$.

Consider in the complex plane a loop $\sigma_0$ moving from the origin, going through $c - \epsilon$, turning positively around $c$ and finally going back to 0. See Figure 1.

The function $F(x)$ can be continued analytically from the origin along $\sigma_0$ and we obtain the new function $\sigma_0 F(x)$.

The 2nd Problem is to express $\sigma_0 F(x)$ explicitly near $x = 0$. It can be stated more precisely as follows.

$\sigma_0 F(x)$ satisfies $(E^*)$. There are $s$ asymptotic series solutions to $(E^*)$

\[ G_k(x) = x^{\rho_k} \left( \xi_{k,0} + \xi_{k,1} x^{\rho_k} + \xi_{k,2} x^{2\rho_k} + \cdots \right) \quad (1 \leq k \leq s) \]

where the coefficient $\xi_{k,0}$ in the 1st term is given by

\[ \xi_{k,0}(\alpha', \alpha) = \frac{\prod_{j=1}^{r} \Gamma(\alpha_j' + \beta_j' \rho_k)}{\Gamma(\rho_k + 1) \prod_{j=1}^{s} \Gamma(\alpha_j + \beta_j \rho_k)} \]

for $\rho_k = -\alpha_k + 1 - \beta_k$.

In fact we fix $k$, $1 \leq k \leq s$. Suppose the asymptotic series

\[ G(x) = x^{\rho_k} \left( \xi_0 + \xi_1 x^{\rho_k} + \xi_2 x^{2\rho_k} + \cdots \right) \quad (1 \geq k \geq s) \]

satisfies $(E^*)$, then $\xi_0(\alpha', \alpha)$ satisfies

\[ T_{\alpha_j'} \xi_0(\alpha', \alpha) = (\alpha_j + \beta_j \rho_k) \xi_0(\alpha', \alpha) \quad (1 \leq j \leq r) \]

\[ \xi_0(\alpha', \alpha) = (\alpha + \beta \rho_k) T_{\alpha_1} \xi_0(\alpha', \alpha) \quad (1 \leq j \leq s, j \neq k) \]

\[ \rho_k \xi_0(\alpha', \alpha) = T_{\alpha_1'} \cdots T_{\alpha_j'} T_{\alpha_k} \cdots T_{\alpha_s} \xi_0(\alpha', \alpha) \]

Hence $\xi_0(\alpha', \alpha)$ can be expressed as

\[ \xi_0(\alpha', \alpha) = \xi_{k,0}(\alpha', \alpha) \cdot U_k \]

where $U_k$ denotes an arbitrary periodic function with the period 1 with respect to each variable $\rho_k$, $\alpha_j' + \beta_j' \rho_k$ ($1 \leq j \leq r$), $\alpha_j + \beta_j \rho_k$ ($1 \leq j \leq s, j \neq k$) $\xi_0$ are then determined uniquely by the relations

\[ \xi_0(\alpha', \alpha) = T_{\alpha_k} \xi_0(\alpha', \alpha) \]

We denote by $U_k[G_k](x)$ the asymptotic series (1.13) thus defined.
In the same way there exists the asymptotic series solution $U_0[F](x)$ satisfying (E*)

\begin{equation}
U_0[F](x) = \xi_{0,0} + \xi_{0,1}x + \xi_{0,2}x^2 + \cdots
\end{equation}

such that

\begin{equation}
\xi_{0,0} = U_0(\alpha', \alpha) \cdot F(0)
\end{equation}

where $U_0(\alpha', \alpha)$ denotes a periodic function of $(\alpha', \alpha)$ with the period 1 with respect to each variable $\alpha'_1, \cdots \alpha'_r, \alpha_1, \cdots \alpha_s$.

**Proposition 1** (See [7]). Every asymptotic series solution at the origin $x = 0$ to the equations (E*) can be expressed as the sum

\begin{equation}
U_0[F](x) + \sum_{k=1}^{s} U_k[G_k](x)
\end{equation}

as asymptotic series for suitable periodic functions $U_0$ and $U_k$.

In particular $W(x)$ and $\sigma_0F(x)$ can be expressed in the form (1.21).

The 2nd Problem is to determine explicitly these periodic functions for $\sigma_0F(x)$.

### 2. Simple Cases

In this section we consider two simple cases where $r = 2, s = 0$, and $r = s = 1$ and solve the Problems stated in the Introduction.

**Case 1.** $(r = 2, s = 0)$.

$F(x)$ has the form

\[
F(x) = \sum_{\nu=0}^{\infty} \frac{\Gamma(\alpha'_1 + \beta'_1 \nu) \Gamma(\alpha'_2 + \beta'_2 \nu)}{\nu!} x^\nu
\]

with $\beta'_1 + \beta'_2 = 1, \beta'_1 > 0, \beta'_2 > 0$. Then $\gamma = \alpha'_1 + \alpha'_2, \delta = -\gamma + \frac{1}{2}$ and $c = \beta'_1 - \beta'_2 > 1$. $F$ has an integral representation

\begin{equation}
F(x) = \Gamma(\gamma) \int_0^\infty u^{\alpha'_1 - 1} (1 + u - u^{\beta'_1} x)^{-\gamma} du
\end{equation}

This has a definite meaning provided $0 < \alpha' < \gamma$, otherwise the integral should be regarded as a finite part of divergent integrals or it should be done on a suitable loop avoiding $u = 0$ or $u = \infty$.

The quasi algebraic equation

\begin{equation}
1 + u - u^{\beta'_1} x = 0
\end{equation}

has a countable number of power series solutions $u_l$ ($l \in \mathbb{Z}$) at $x = 0$

\begin{equation}
u_l = -1 + e^{\pi(1+2l)\beta'_1} x + \cdots .
\end{equation}
When \( x \) moves from 0 to \( c \), the only 2 solutions \( u_0, u_{-1} \) (\( u_{-1} \) is the complex conjugate of \( u_0 \)) meet each other at \( x = c \). Furthermore, Im \( u_0 > 0 \), Im \( u_{-1} < 0 \) for \( 0 < x < c \). See Figure 2.

We denote by \( C_0 \) the interval \([0, \infty)\) which is positively oriented, and by \( L_0 \) the oriented segment from \( u_0 \) to \( u_{-1} \). The intersection number between them is equal to \(-1\),

\[
I(C_0, L_0) = -1
\]

Consider also the integral \( W(x) \) over \( L_0 \)

\[
W(x) = \Gamma(\gamma) \int_{L_0} u^{\alpha'-1}(1 + u - u^{\beta_i}x)^{-\gamma} du
\]

which is well defined provided \( \gamma < 1 \), otherwise should be suitably regularized.

\( C_0 \) and \( L_0 \) are twisted cycles for the multivalued function \( u^{\alpha'-1}(1 + u - u^{\beta_i}x)^{-\gamma} \).

When \( x \) moves along the loop \( \sigma_0 \) from \( c - \varepsilon \) to itself, they both can be deformed in an isotopic way as follows

\[
\sigma_0 : \left\{ \begin{array}{l}
C_0 \to C_0 + (1 - e^{-2\pi i\gamma})L_0 \\
L_0 \to -e^{-2\pi i\gamma}L_0
\end{array} \right.
\]

This implies

\[
\sigma_0 F(x) = F(x) + (1 - e^{-2\pi i\gamma})W(x)
\]

\[
\sigma_0 W(x) = -e^{-2\pi i\gamma}W(x)
\]

\( W(x) \) has the power series expansion near \( x = c \) as follows

\[
W(x) = (c - x)^{\delta} \left\{ a_0^* + a_1^*(c - x) + a_2^*(c - x)^2 + \cdots \right\}
\]

In fact for \( x < c \),

\[
\frac{1 + u}{u^{\beta_i}} = c - x + \frac{1}{2} \beta_1' - \beta_2' - 1 \beta_2'^2 u - \frac{\beta_2'}{\beta_2'^2} + \cdots
\]

and

\[
u_0 = \frac{\beta_1'}{\beta_2'} + i\sqrt{2(c - x)} + \cdots
\]

\[
u_{-1} = \frac{\beta_1'}{\beta_2'} - i\sqrt{2(c - x)} + \cdots
\]

By using (2.6), (2.9), (2.10) and Euler’s formula for the Beta integral, we can evaluate \( a_0^* \) as follows.

**Lemma 1.**

\[
a_0^* = -i\sqrt{2\pi \beta_1' \beta_2'^3 \Gamma(1 - \gamma) \Gamma(\gamma)} \frac{\Gamma(\frac{3}{2} - \gamma)}{\Gamma(\frac{3}{2} - \gamma)}
\]
for
\[
\begin{align*}
\epsilon_1' &= \alpha_1' + \beta_1' \delta - \frac{1}{2} \\
\epsilon_2' &= \alpha_2' + \beta_2' \delta - \frac{1}{2}
\end{align*}
\]
respectively.

From (1.8), (1.9) and (2.8), we have
\[
\sigma_0 F(x) - F(x) = (e^{2\pi i \delta} - 1)(c - x)\delta \{a_0 + a_1(c - x) + \cdots\} \\
= (1 - e^{-2\pi i \gamma})(c - x)\delta \{a_0^* + a_1^*(c - x) + \cdots\}
\]
whence the equalities hold
\[
a_\nu = \frac{1 - e^{-2\pi i \gamma}}{e^{2\pi i \delta} - 1} a_\nu^* \quad (\nu \geq 0)
\]
in particular
\[
a_0 = -i \tan \pi \gamma \cdot a_0^*
\]

Therefore

**Proposition 2.** $a_0$ is evaluated as
\[
a_0 = -\sqrt{2\pi} \tan \pi \gamma \cdot \beta_1' \epsilon_1' \beta_2' \epsilon_2' \cdot \frac{\Gamma(1 - \gamma)\Gamma(\gamma)}{\Gamma\left(\frac{3}{2} - \gamma\right)}
\]
where $\epsilon_1'$, $\epsilon_2'$ are defined by (2.12).

Now let us solve the 2nd Problem. In view of (2.7), we have only to express $W(x)$ at $x = 0$.

As $x$ decreases to $x = 0$, the cycle $\mathcal{L}_0$ is deformed smoothly so that $W(x)$ becomes holomorphic at $x = 0$. There the cycle $\mathcal{L}_0$ is identified with a loop starting from and ending in $x = -1$, and turning around 0 in a positive direction. Hence we have
\[
W(0) = \Gamma(\gamma) \int_{\mathcal{L}_0} u^{\alpha_1'-1}(1 + u)^{-\gamma}du
\]
\[
= -2i\pi \frac{\Gamma(\gamma)\Gamma(1 - \gamma)}{\Gamma(1 - \alpha_1')\Gamma(1 - \alpha_2')}
\]
\[
= -2i \frac{\sin \pi \alpha_1' \sin \pi \alpha_2'}{\sin \pi \gamma} F(0)
\]
which implies
\[
W(x) = U_0[F](x)
\]
for $U_0(\alpha_1', \alpha_2') = -2i \frac{\sin \pi \alpha_1' \sin \pi \alpha_2'}{\sin \pi \gamma}$. 

We now define the functions $F_{l_1', l_2'}(x)$ for $(l_1', l_2') \in \mathbb{Z}^2$, as
\begin{equation}
F_{l_1', l_2'}(x) = \{ e^{2\pi i (l_1' \alpha_1' + l_2' \alpha_2')} [F] \}(x) = e^{2\pi i (l_1' \alpha_1' + l_2' \alpha_2')} \cdot F(e^{2\pi i (l_1' \beta_1' + l_2' \beta_2')} x).
\end{equation}

Then it is obvious that $F(x) = F_{0,0}(x)$ and $F_{l_1', l_2'}(x) = e^{2\pi i \gamma} F_{l_1'-l_2',0}(x)$.

From (2.15) and (2.16) we have
\begin{equation}
\text{PROPOSITION 3.}
\end{equation}
\begin{equation}
W(x) = \frac{1}{1 - e^{2\pi i \gamma}} \{ (e^{2\pi i \gamma} + 1) F_{0,0}(x) - F_{1,0}(x) - e^{2\pi i \gamma} F_{-1,0}(x) \}
\end{equation}

As a consequence of (2.7), we have the monodromy formula
\begin{equation}
\text{COROLLARY 1.}
\end{equation}
\begin{equation}
\sigma_0 F_{0,0}(x) = -e^{-2\pi i \gamma} F_{0,0}(x) + e^{-2\pi i \gamma} F_{1,0}(x) + F_{-1,0}(x)
\end{equation}
\begin{equation}
\sigma_0 F_{\mu,0}(x) = F_{\mu,0}(x) (\mu \neq 0).
\end{equation}

Similarly, we can also consider the analytic continuation of the functions
\begin{equation}
F_{l_1,0}(x)
\end{equation}
along a loop $\sigma_0 (\infty < \nu < \infty)$ corresponding to the movement from
the origin to itself turning around $e^{2\pi i \beta_1 \nu}$.

The monodromy formula for $\sigma_0$ is given by the simple shift of the indices $l$ to $l + \nu$ as follows
\begin{equation}
\text{(2.18)}
\end{equation}
\begin{equation}
\sigma_0 F_{0,0}(x) = -e^{-2\pi i \gamma} F_{0,0}(x) + e^{-2\pi i \gamma} F_{\nu+1,0}(x) + F_{\nu-1,0}(x)
\end{equation}
\begin{equation}
\sigma_0 F_{\mu,0}(x) = F_{\mu,0}(x) (\mu \neq \nu).
\end{equation}

This transformation is nothing else than a Burau type representation corresponding to an infinite number of strands.

Case 2. ($r = s = 1$).

Consider the function
\begin{equation}
F(x) = \sum_{\nu=0}^{\infty} \frac{\Gamma(\alpha_1' + \beta_1' \nu)}{\Gamma(\alpha_1 + \beta_1 \nu)} x^\nu
\end{equation}
with $\beta_1' = \beta_1 + 1$, $\beta_1 > 0$. Then $\gamma = \alpha_1' + 1 - \alpha_1$, $\delta = -\gamma + 1/2$ and $c = \beta_1' - \beta_1 - \beta_1' < 1$.

The integral formula for $F(x)$ can be expressed as follows. For $0 < x < c$, the quasi algebraic equation with respect to $\nu$
\begin{equation}
(2.20)
1 - x + v^{\beta_1'} x = 0
\end{equation}
has 2 real solutions $v_0$ and $v_0^*$ such that $1 < v_0 < v_0^*$.

We denote by $L_0$ a loop moving from 0 to itself, passing through the open interval $(v_0, v_0^*)$, and by $L_0$ the closed interval $[v_0, v_0^*]$ as in Figure 4.
Then we have

\[ F(x) = \frac{\Gamma(\gamma)}{2\pi i} \int_{C_0} v^{\alpha'_1 - 1} (v - 1 - \nu^{\beta'_1} x)^{-\gamma} dv \]

(2.21)

\[ = \frac{1}{\Gamma(1 - \gamma)} \int_0^{v_0} v^{\alpha'_1 - 1} (1 - v + \nu^{\beta'_1} x)^{-\gamma} dv \]

where the function \( v - 1 - \nu^{\beta'_1} x \) is taken to be positive in \((v_0, v_0^*)\).

We also put

\[ W(x) = \Gamma(\gamma) \int_{L_0} v^{\alpha'_1 - 1} (v - 1 - \nu^{\beta'_1} x)^{-\gamma} dv \]

(2.22)

Remark that \( v_0(+) = 1 \) and \( v_0^*(+) = +\infty \). When \( x \) increases from 0 to \( c \), \( v_0 \) and \( v_0^* \) meet each other. The intersection number between \( C_0 \) and \( L_0 \) is equal to \(-1\), i.e., we have the same formula as (2.4).

Near \( x = c \), \( F(x) \) and \( W(x) \) have the expansions (1.8)-(1.9) and (2.8), respectively.

We have

**Lemma 2.**

\[ a_0^* = \sqrt{2\pi \beta_1^{e_1}} \frac{e_1 \Gamma(1 - \gamma) \Gamma(\gamma)}{\Gamma(3/2 - \gamma)} \]

(2.23)

where

\[ e_1 = \alpha'_1 + \beta'_1 \delta - \frac{1}{2} \]

(2.24)

and

\[ e_1 = -\alpha_1 - \beta_1 \delta + \frac{1}{2} \]

In fact we have the expansions near \( x = c \) and for \( x < c \)

\[ \frac{v - 1}{\nu^{\beta'_1}} = c - x - \frac{1}{2} \beta'_1^{e_1 - 1} \beta_1^{e_1 + 2} (u - \frac{\beta'_1}{\beta_1})^2 + \ldots \]

(2.25)

and

\[ v_0 = \frac{\beta'_1}{\beta_1} - \sqrt{2(c - x)} + \ldots \]

(2.26)

\[ v_0^* = \frac{\beta'_1}{\beta_1} + \sqrt{2(c - x)} + \ldots \]

Then the proof is similar to Lemma 1.

Since (2.6) holds, we have the transformation formula in view of (2.22),

\[ \sigma_0 F(x) = F(x) + \frac{(1 - e^{-2\pi i \gamma})}{2\pi i} W(x) \]

(2.27)

\[ \sigma_0 W(x) = -e^{-2\pi i \gamma} W(x) \]
so that

\begin{equation}
(2.28) \quad a_0 = -\frac{\tan \pi \gamma}{2\pi} \cdot a_0^*
\end{equation}

Combining (2.25) and (2.30), we have

**Proposition 4.**

\begin{equation}
(2.29) \quad a_0 = -\frac{1}{\sqrt{2\pi}} \beta_1 \beta_1^* \beta_1^* \Gamma(1 - \gamma) \Gamma(\gamma) \tan \pi \gamma.
\end{equation}

The behaviour of \( W(x) \) at \( x = 0 \) is a little more complicated than Case 1. As is seen from (2.22), when \( x \) approaches 0, \( W(x) \) has the singularity at \( x = 0 \). To make clear that singularity, we also define the function

\begin{equation}
(2.30) \quad G^*(x) = \Gamma(\gamma) \int_0^\infty v^{\alpha_1 - 1}(1 - v + v^{\beta_1}x)^{-\gamma}dv
\end{equation}

where the function \( 1 - v + v^{\beta_1}x \) should be taken positive in the interval \((v_0^*, \infty)\).

We assume for simplicity that \( \alpha_1' - \beta_1^* \gamma < 0 \), so that the RHS of (2.30) converges absolutely. Since \( v_0^*(x) \) has a Puiseaux expansion at \( x = 0 \) as

\[ v_0^*(x) = x^{-\frac{1}{\beta_1}} \{ 1 + O(x^{-\frac{1}{\beta_1}}) \}, \]

by making a careful estimate of (2.30), one can show that \( G^*(x) \) has the asymptotic expansion

\begin{equation}
(2.31) \quad G^*(x) = x^{\rho_1} (\xi_0^* + \xi_1^* x^{\frac{1}{\beta_1}} + \xi_2^* x^{\frac{2}{\beta_1}} + \cdots) \quad (1 \leq k \leq s)
\end{equation}

for

\begin{equation}
(2.32) \quad \rho_1 = \frac{-\alpha_1 + 1}{\beta_1}, \quad \xi_0^* = \frac{\pi \Gamma(\rho_1 + \gamma)}{\beta_1 \sin \pi \gamma \Gamma(\rho_1 + 1)}
\end{equation}

We now state

**Proposition 5.** \( W(x) \) **has the asymptotic expansion at** \( x = 0 \) **as**

\begin{equation}
(2.33) \quad W(x) = x^{\rho_1} (\zeta_{1,0} + \zeta_{1,1} x^{\frac{1}{\beta_1}} + \zeta_{1,2} x^{\frac{2}{\beta_1}} + \cdots) + \zeta_{0,0} + \zeta_{0,1} x + \zeta_{0,2} x^2 + \cdots
\end{equation}

where

\begin{equation}
(2.34) \quad \zeta_{1,0} = -\frac{\sin \pi (\rho_1 + \gamma)}{\sin \pi \rho_1} \xi_0^*
\end{equation}

\begin{equation}
(2.35) \quad \zeta_{0,0} = \frac{\pi \Gamma(1 - \alpha_1)}{\sin \pi \gamma \Gamma(1 - \alpha_1')} = \frac{\pi \sin \pi \alpha_1'}{\sin \pi \gamma \sin \pi \alpha_1} F(0)
\end{equation}

respectively.
Indeed, we fix a sufficiently large positive number \( h \). Then the RHS of (2.23) can be divided into two parts

\[
W(x) = \Gamma(\gamma) \int_{v_0}^{h} \{ \} dv + \Gamma(\gamma) \int_{h}^{v_0} \{ \} dv \tag{2.36}
\]

To evaluate the 2nd term, we make a change of variables \( \tilde{v} = \frac{v - 1}{v_{\beta_1}} \) and put \( \tilde{h} = \frac{h - 1}{h_{\beta_1}} \).

Then \( v \) is expanded in Laurent series of \( \tilde{v}^{\beta_1} \) as

\[
v = \tilde{v}^{-\frac{1}{\beta_1}} \varphi(\tilde{v}) \quad \text{and} \quad \varphi(\tilde{v}) = 1 + \varphi_1 \tilde{v}^{-\frac{1}{\beta_1}} + \varphi_2 \tilde{v}^{-\frac{2}{\beta_1}} + \ldots
\]

so that we have

\[
\tilde{h} \int_{\tilde{v}}^{v_0} v^\alpha_{\beta_1}^{-1} \left( v - 1 - v_{\beta_1} \right)^{-\gamma} dv
\]

\[
= \tilde{h} \int_{x}^{\tilde{h}} \left\{ \tilde{v} \varphi(\tilde{v}) \right\}^{-\alpha_1 + \beta_1} \frac{1}{\beta_1} \left( \frac{1}{\varphi(\tilde{v})} - \varphi'(\tilde{v}) \right) d\tilde{v}
\]

\[
= \tilde{h} \int x^{\rho_1 + \gamma}(\tilde{v} - x)^{-\gamma} \frac{1}{\beta_1} (1 + c_1 \tilde{v}^{-\frac{1}{\beta_1}} + c_2 \tilde{v}^{-\frac{2}{\beta_1}} + \ldots) d\tilde{v}
\]

for suitable constants \( c_1, c_2, c_3, \ldots \). The last part of (2.37) has an asymptotic form (2.33).

The first term in the last part of (2.37) can be described as

\[
= \frac{1}{\tilde{h}^{\rho_1 + \gamma}} \frac{1}{\beta_1} d\tilde{v}
\]

\[
= \frac{1}{e^{2\pi i \rho_1}} - 1 \int_{x}^{\tilde{h}} \tilde{v}^{\rho_1 + \gamma}(\tilde{v} - x)^{-\gamma} \frac{1}{\beta_1} d\tilde{v}
\]

\[
- \frac{\sin \pi \rho_1 + \gamma}{\sin \pi \rho_1} \int_{0}^{x} \tilde{v}^{\rho_1 + \gamma}(\tilde{v} - x)^{-\gamma} \frac{1}{\beta_1} d\tilde{v}
\]

where \( a \) denotes a loop starting from and ending in \( \tilde{h} \) and turning around 0 positively. The 1st term is therefore holomorphic at \( x = 0 \), while the second term equals, due to Euler formula,

\[
- \frac{\sin \pi (\rho_1 + \gamma)}{\sin \pi \rho_1} \cdot \frac{\Gamma(\rho_1 + \gamma) \Gamma(1 - \gamma)}{\Gamma(\rho_1 + 1) \beta_1} x^{\rho_1}
\]

Hence (2.33) follows.
Let us prove (2.35). The 2nd term in the RHS of (2.33) has the integral representation

\[ \zeta_{0,0} + \zeta_{0,1}x + \zeta_{0,2}x^2 + \cdots \]

\[ = \Gamma(\gamma) \int_{i0}^{h} v^{\alpha_1'-1}(v - 1 - v\beta_1 x)^{-\gamma} dv \]

\[ + \Gamma(\gamma) \int_{a}^{b} \frac{1}{\beta_1 \tilde{v}} \left( \frac{1}{e^{2\pi i \rho_1} - 1} + \frac{c_1 \tilde{v}^{-\beta_1}}{e^{2\pi i (\rho_1 - \beta_1)\tilde{v}} - 1} + \frac{c_2 \tilde{v}^{-\beta_1}}{e^{2\pi i (\rho_1 - \beta_1)\tilde{v}} - 1} + \cdots \right) d\tilde{v} \]  

(2.39)

When \( x \) tends to 0, the 2nd term in the RHS reduces to

\[ \Gamma(\gamma) \int_{h}^{\infty} v^{\alpha_1'-1}(v - 1 - v\beta_1 x)^{-\gamma} dv \]

provided \( \alpha_1' - \beta_1' \gamma < 0 \), in which case we have

\[ \zeta_{0,0} = \Gamma(\gamma) \int_{1}^{\infty} v^{\alpha_1'-1}(v - 1)^{-\gamma} dv \]

which is equal to (2.35).

In case where \( \alpha_1' - \beta_1' \gamma \geq 0 \), since \( \zeta_{0,0} \) can be given as a regularized integral, one can show that (2.35) is still true.

Proposition 5 has thus been proved.

As a consequence, the formula (1.23) can be written as follows

\[ W(x) = U_0[F](x) + U_1[G_1](x) \]

where \( U_0 = \frac{\pi \sin \pi \alpha_1'}{\sin \pi \gamma \sin \pi \alpha_1} \) and \( U_1 = -\frac{\pi \sin \pi (\rho_1 + \gamma)}{\beta_1 \sin \pi \gamma \sin \pi \rho_1} \).

\( \sigma_0 F(x) \) is expressed in the form (2.27).

3. Theorem and Conjecture in General Case

Now we assume \( r, s \geq 1 \) are general. For convenience we use the notations

\[ \beta_{r+1}' = -\beta_1, \ldots, \beta_{r+s}' = -\beta_s, \]

\[ \alpha_{r+1}' = 1 - \alpha_1, \ldots, \alpha_{r+s}' = 1 - \alpha_s. \]
Let the functions $\Phi(v, x)$ be defined by

\begin{equation}
\Phi(v, x) = v_1^{\alpha_1-1} \cdots v_n^{\alpha_n-1} \{v_1 + \cdots + v_n + 1 - g(v)x\}^{-\gamma}
\end{equation}

or

\begin{equation}
\Phi(v, x) = v_1^{\alpha_1-1} \cdots v_n^{\alpha_n-1} \{v_1 + \cdots + v_r - 1 - v_{r+1} - \cdots - v_n - g(v)x\}^{-\gamma}
\end{equation}

for $r \geq 2$, $s = 0$, or $r \geq 1$, $s \geq 1$, where $g(v)$ denotes the power $v_1^{\beta_1} \cdots v_n^{\beta_n}$.

We want to find a pair of $n$ dimensional twisted cycles $C_0$ and $L_0$ depending on $x$, $x \neq c$ continuously, such that

\begin{equation}
F(x) = \frac{\Gamma(\gamma)}{(2\pi i)^s} \int_{C_0} \Phi(v, x) dv_1 \wedge \cdots \wedge dv_n
\end{equation}

\begin{equation}
W(x) = \Gamma(\gamma) \int_{L_0} \Phi(v, x) dv_1 \wedge \cdots \wedge dv_n
\end{equation}

satisfy the following properties.

(i) The intersection number between $C_0$ and $L_0$ as chains is equal to $(-1)^n$, i.e.,

\begin{equation}
I(C_0, L_0) = (-1)^n
\end{equation}

(ii) When $x = 0$, the RHS of (3.2) and (3.3) reduce to

\begin{equation}
\frac{1}{(2\pi i)^s} \int_{C_0} \Phi(v, 0) dv_1 \wedge \cdots \wedge dv_n = \frac{\Gamma(\alpha'_1) \cdots \Gamma(\alpha'_r)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_s) \Gamma(\gamma)}
\end{equation}

\begin{equation}
\int_{L_0} \Phi(v, 0) dv_1 \wedge \cdots \wedge dv_n = (-2\pi i)^{r-1} \frac{\Gamma(1 - \alpha_1) \cdots \Gamma(1 - \alpha_s) \Gamma(1 - \gamma)}{\Gamma(1 - \alpha'_1) \cdots \Gamma(1 - \alpha'_r)}
\end{equation}

We denote the LHS of (3.5) and (3.6) by $J_1(\alpha', \alpha)$ and $J_2(\alpha', \alpha)$, respectively.

(iii) The generalized Picard-Lefschetz transformation for $\sigma_0$ (due to F. Pham, see [13, 14]) is expressed as

\begin{equation}
\sigma_0 : C_0 \to C_0 + (1 - e^{-2\pi i\gamma})L_0
\end{equation}

\begin{equation}
L_0 \to (-1)^n e^{-2\pi i\gamma} L_0
\end{equation}

so that we have

\begin{equation}
\sigma_0 F(x) = F(x) + \frac{1 - e^{-2\pi i\gamma}}{(2\pi i)^s} W(x)
\end{equation}

\begin{equation}
\sigma_0 W(x) = (-1)^n e^{-2\pi i\gamma} W(x)
\end{equation}

The cycles $C_0$, $L_0$ can be constructed in a geometric way in the following manner.
Let \( \omega'_1, \ldots, \omega'_r, \omega_1 \ldots \omega_s \) be positive integers, such that \( \omega_0 = \sum_{j=1}^{r} \omega'_j > 0 \) \((s = 0)\) and \( \omega_0 = \sum_{j=1}^{r} \omega'_j - \sum_{j=1}^{s} \omega_j \) \((s \geq 1)\) are both positive. We fix complex numbers \( \alpha'_j, \alpha_j \) and \( \gamma = \sum_{j=1}^{r} \alpha'_j + s - \sum_{j=1}^{s} \alpha_j \). We want to apply Morse theory to the integrals (3.2), (3.3), (3.5) and (3.6). According as \( x \neq 0 \) or \( x = 0 \) we consider the level function \( H(x) \) or \( H(0) \) in the directions \( \pm(\omega', \omega) = \pm(\omega'_1, \ldots, \omega'_r, \omega_1 \ldots \omega_s) \)

\[
H(x) = \sum_{j=1}^{r} \omega'_j \log |v_j| - \omega_0 \log |f(v, x)| \quad (s = 0)
\]

\[
= \sum_{j=1}^{r} \omega'_j \log |v_j| - \sum_{j=1}^{s-1} \omega_j \log |v_{j+r}| - \omega_0 \log |f(v, x)| \quad (s \geq 1)
\]

where \( f(v, x) \) denotes

\[
f(v, x) = \begin{cases} v_1 + \cdots + v_n + 1 - g(v)x & (s = 0) \\ v_1 + \cdots + v_r - 1 - v_{r+1} - \cdots - v_n - g(v)x & (s \geq 1) \end{cases}
\]

respectively.

For a fixed \( x \) such that \( 0 \leq x < c \), we can define the gradient vector field \( V(x) \) in the space \( \mathbb{C}^n - \cup_{j=1}^{n} \{v_j = 0\} \cup \{f(v, x) = 0\} \)

\[
V(x) = \text{grad}_v H(x)
\]

\[
= \left( \frac{\partial H}{\partial \text{Re} v_1}, \ldots, \frac{\partial H}{\partial \text{Re} v_n}, \frac{\partial H}{\partial \text{Im} v_1}, \ldots, \frac{\partial H}{\partial \text{Im} v_n} \right)
\]

In particular \( V(0) \) has the unique critical point \( v = v(c) \)

\[
v(c) : \begin{cases} v_j = \omega'_j (1 \leq j \leq r - 1) & \text{for } s = 0 \\ v_j = \omega'^{j} s_j (1 \leq j \leq r), v_{j+r} = \omega^{j}_j (1 \leq j \leq s - 1) & \text{for } s \geq 1 \end{cases}
\]

such that \( f(v(c), 0) = \frac{\omega_0}{\omega'_r} (s = 0) \) or \( \frac{\omega_0}{\omega'_s} (s \geq 1) \) which is positive.

Let \( C_0 \) and \( L_0 \) be the cycles constructed in the complement \( \mathbb{C}^n - \bigcup_{j=1}^{n} \{v_j = 0\} \cup \{f(v, 0) = 0\} \) such that in a neighbourhood of the critical point \( v(c) \) they are stable and unstable with respect to the vector field \( V(0) \). Then the pair \( C_0 \) and \( L_0 \) satisfies (3.4) for \( x = 0 \).

We are now in a position to prove (3.5) and (3.6).
The LHS of (3.5) and (3.6) both satisfy the difference equations

\[ T_{\alpha_k} J_1(\alpha', \alpha) = \frac{\alpha'_k}{\gamma} J_1(\alpha', \alpha) \]
\[ T_{\alpha_k} J_1(\alpha', \alpha) = \frac{\gamma - 1}{\alpha_k} J_1(\alpha', \alpha) \]

(3.12)

When \( N \) tends to \(+\infty\) in the direction \((\omega', \omega)\), we see by saddle point method that \( J_1(\alpha', \alpha) \) has the asymptotic

\[ J_1(\alpha', \alpha) \sim (2\pi)^{\frac{n}{2}} N^{-\frac{n}{2}} \omega_1^{\alpha_1} \cdots \omega_r^{\alpha_r} \omega_1^{\alpha_1} \cdots \omega_s^{\alpha_s} + \frac{1}{2} \omega_0^{\gamma + \frac{1}{2}} \]

(3.13)

The RHS of (3.5) also satisfies (3.12) and has the same asymptotic as (3.13), whence they must coincide.

On the contrary, when \( N \) tends to \(+\infty\) in the opposite direction \((-\omega', -\omega)\), \( J_2(\alpha', \alpha) \) and the RHS of (3.6) satisfy (3.12) and have the same asymptotic as the RHS of (3.13), whence these two must coincide. Property (ii) has thus been shown.

As \( x \) moves from 0 to \( c - \epsilon \) along the real axis, the cycles \( C_0 \) and \( L_0 \) can be deformed in an isotopic way such that they are stable and unstable cycles as to the vector field \( V(x) \) respectively for each \( x \). They always satisfy (3.4). When \( x \) approaches \( c \), the cycle \( L_0 \) vanishes. We want to show from the formula (3.3), that \( W(x) \) has the expansion (2.8) and evaluate the 1st coefficient \( a_0^* \).

When \( x = c \), the critical point for the function \( H(x) \) coincides with the point \( \left( \frac{\beta'_1}{\beta'_r}, \cdots, \frac{\beta'_{s-1}}{\beta'_r} \right) \) (s = 0), \( \left( \frac{\beta'_1}{\beta'_r}, \cdots, \frac{\beta'_{r-1}}{\beta'_r}, \cdots, \frac{\beta'_{s-1}}{\beta'_r} \right) \) (s \( \geq 1 \)).

Near the critical point there exist local coordinates \((w_1, \ldots, w_n)\) such that

\[ f(v, x) = c - x + w_1^2 + \cdots + w_{r-1}^2 - w_r^2 - \cdots - w_n^2. \]

(3.14)

Near \( x = c \) the cycle \( L_0 \) is homologous to the \( n \) dimensional disk defined by

\[ D = \{ (t_1, \ldots, t_n) \in \mathbb{R}^n; \ c - x \geq t_1^2 + \cdots + t_n^2 \} \]

for \( w_1 = -it_1, \ldots, w_{r-1} = -it_{r-1}, w_r = t_r, \ldots, w_n = t_n \). It shrinks to a point and therefore vanishes at \( x = c \).

The integral (3.3) is rewritten as

\[ W(x) = \Gamma(\gamma) \int_D v_1^{\alpha'_1 \gamma - 1} \cdots v_n^{\alpha'_n \gamma - 1} \]
\[ (c - x - t_1^2 - \cdots - t_n^2)^{-\gamma} \frac{\partial(v_1, \ldots, v_n)}{\partial(t_1, \ldots, t_n)} dt_1 \wedge \cdots \wedge dt_n \]

(3.16)
where \( \frac{\partial(v_1, \ldots, v_n)}{\partial(t_1, \ldots, t_n)} \) denotes the Jacobian of the variables \( v_1, \ldots, v_n \) relative to \( t_1, \ldots, t_n \).

The Hessian of the quotient \( f(v, x)/g(v) \) with respect to \( v_1, \ldots, v_n \) at \( v(c) \) is equal to

\[
(-1)^s e^n \frac{\beta_{s-1}}{\beta_1 \ldots \beta_r \beta_1 \ldots \beta_s}
\]

which implies

(3.17) \[ \left[ \frac{\partial(v_1, \ldots, v_n)}{\partial(t_1, \ldots, t_n)} \right]_{v=v(c)} = e^{-\frac{\gamma}{2}} \sqrt{\beta_1 \ldots \beta_r \beta_1 \ldots \beta_s} \frac{(-i)^{r-1}}{\beta_{s-1}} \]

The formulae for \( a_0 \) and \( a_0^* \) are evaluated from (3.16) and (3.17) as follows.

**Theorem 1.** \( W(x) \) has the expansion (2.8) such that

(3.18) \[ a_0^* = (-i)^{r-1}(2\pi)^{\gamma} \frac{\Gamma(1 - \gamma)\Gamma(\gamma)}{\Gamma(\delta + 1)} \prod_{j=1}^n \beta_j e_j' \]

where

(3.19) \[ e_j' = \alpha_j' + \beta_j' \delta - \frac{1}{2} \]

(Remark that we have put \( \alpha_j = 1 - \alpha_{j+r}, \beta_j = -\beta_{j+r} \) for \( 1 \leq j \leq s \), and

\( a_0 = \frac{(1 - e^{-2\pi i})}{(2\pi i)^{n} (e^{2\pi i\delta} - 1)} a_0^* \)

As for the 2nd Problem, as a generalization of the formulae (2.17) and (2.40) one may make the following

**Conjecture 1.** \( W(x) \) has the asymptotic expansion (1.23)

(3.20) \[ W(x) = \zeta_{0,0} + \zeta_{0,1} x + \zeta_{0,2} x^2 \ldots \]

\[ + \sum_{k=1}^s \zeta_{k,0} + \zeta_{k,1} x^{\frac{1}{k}} + \zeta_{k,2} x^{\frac{2}{k}} \ldots \]

at the origin such that

(3.21) \[ U_0(\alpha', \alpha) = (-2\pi i)^{r-1} \frac{\pi}{\sin \pi \gamma} \prod_{j=1}^r \sin \pi \alpha_j' \frac{\pi}{\sin \pi \alpha_j} \]

\[ \prod_{j=1}^s \frac{\pi}{\sin \pi \alpha_j} \]

\[ U_k(\rho_k, \{\alpha_j' + \beta_j' \rho_k\}_{j=1}^r, \{\alpha_j + \beta_j \rho_k\}_{j=1}^s) \]

(3.22) \[ = \frac{1}{\beta_k \sin \pi \rho_k \sin \pi \gamma} \prod_{j=1}^r \sin \pi (\alpha_j' + \beta_j' \rho_k) \frac{\pi}{\sin \pi \alpha_j + \beta_j \rho_k} \]

\[ \sigma_0 F(x) \] is expressed in the form (3.8).
In the sequel we show in a few special cases that the formula (3.21) and (3.22) are true.

Assume first $s = 0$. For $0 < x < c$, $C_0$ can be realized as the real domain
\[ v_1 \geq 0, \ldots, v_n \geq 0, \]
while $C_0$ lies in the totally imaginary space such that its intersection with the real space $\mathbb{R}^n$ consists of only the critical point of $V(x)$. As $x$ decreases from $c - \epsilon$ to 0, $C_0$ remains compact and moves smoothly. Hence $W(x)$ is holomorphic at the origin, and we get the formula (3.6)
\[ W(0) = \Gamma(\gamma) \int_{\mathcal{L}_0} \ldots \cdot (1 + v_1 + \cdots + v_n)^{-\gamma} dv_1 \wedge \cdots \wedge dv_n \]
\[ = (-2\pi i)^n \frac{\Gamma(\gamma)\Gamma(1 - \gamma)}{\Gamma(1 - \alpha'_1) \cdots \Gamma(1 - \alpha'_n)} = (-2\pi i)^n \frac{\sin \pi \alpha'_1}{\pi} \cdots \frac{\sin \pi \alpha'_n}{\sin \pi \gamma}. \]

This shows Conjecture is true for $s = 0$.

On the other hand, when $r = 1$, for $0 < x < c$, $C_0$ can be realized as a compact real domain defined by
\[ f(v, x) \geq 0 \]
while $C_0$ becomes totally imaginary. We proceed as in having derived (2.36)--(2.39). As $x$ tends to 0, $C_0$ becomes enlarged to the whole $n$-simplex $\Delta$ defined by
\[ v_1 - 1 - v_2 - \cdots - v_n \geq 0, \quad v_2 \geq 0, \ldots, v_n \geq 0 \]
so that we have
\[ \zeta_{0, 0} = \Gamma(\gamma) \int_{v_1 \geq 0, \ldots, v_n \geq 0} \ldots \cdot (1 + v_1 + \cdots + v_n)^{-\gamma} dv_1 \wedge \cdots \wedge dv_n \]
\[ = \Gamma(\gamma) \frac{\Gamma(1 - \alpha'_1) \cdots \Gamma(1 - \alpha'_n)\Gamma(1 - \gamma)}{\Gamma(1 - \alpha'_1)} \]
\[ = \frac{\pi^n \sin \pi \alpha'_1 \cdots \sin \pi \alpha'_n}{\sin \pi \gamma \sin \pi \alpha_1 \cdots \sin \pi \alpha_n} F(0). \]

The boundary of $\Delta$ consists of $n - 1$ dimensional faces $\Delta_{k-1}$ defined by
\[ \Delta_{k-1}(n \geq k \geq 2) : v_k = 0, \quad v_j \geq 0 (j \geq 2, j \neq k), \quad v_1 - 1 - v_2 - \cdots - v_n \geq 0 \]
and
\[ \Delta_0 : v_2 \geq 0 (j \geq 2), \quad v_1 - 1 - v_2 - \cdots - v_n = 0 \]

Let $D_j (n - 1 \geq j \geq 2)$ be the closed set in $\Delta$ which is the composite of all segments connecting $v(c)$ and points in $\Delta_j$ and let $D_n$ be defined by the
inequalities
\begin{align}
   v_1 - v_1(c) - (v_2 - v_2(c)) - \cdots - (v_n - v_n(c)) & \geq 0, \\
v_2 & \geq v_2(c), \cdots v_n \geq v_n(c)
\end{align}
(3.29)
such that we have \( \Delta = \bigcup_{j=0}^{n} D_j \). Then the closure of the complement \( \Delta - [L_0] \)
([L_0] denotes the support of \( L_0 \)) is divided into
\begin{align}
   \Delta - [L_0] = \bigcup_{j=0}^{n} D_j \cap \Delta - [L_0].
\end{align}
(3.30)
(See Figure 6.)

We can construct a twisted cycle \( C_j^* \) such that the support \( C_j^* \) contains \( D_j \cap \Delta - [L_0] \) and that the support of the boundary of \( C_j^* \) denoted by \( \partial C_j^* \) is contained in the set \( \{ f(v, x) = 0 \} \cup \{ v_j = 0 \} \).

We put
\[ \tilde{v}_{k+1} = (v_1 - 1 - v_2 - \cdots - v_n)/g(v) \]
and make a change of variables from \( (v_1, \ldots, v_n) \) into \( (\tilde{v}_{k+1}, v_1, v_2, v_{k+2}, \ldots, v_n) \). One can show that the function \( \tilde{G}_k(x)(1 \leq k \leq s) \) defined by the integral
\begin{align}
   \tilde{G}_k(x) = \Gamma(\gamma) \int_{C_k^*} v_1^{\alpha_1'-1} \cdots v_n^{\alpha_n'-1} f(v, x)^{-\gamma} dv_1 \wedge \cdots \wedge dv_n
\end{align}
(3.31)
has the asymptotic as
\begin{align}
   \tilde{G}_k(x) = x^{\rho_k} \left( \xi_{k,0} + \xi_{k,1} x^{1/\beta_k} + \cdots \right)
\end{align}
(3.32)
To evaluate \( \xi_{k,0}(1 \leq k \leq n-1) \), we first integrate (3.31) over \( C_k^* \) with respect to \( \tilde{v}_{k+1} \) and then with respect to the remaining variables.

Indeed \( \xi_{k,0} \) has the integral representation
\begin{align}
   \xi_{k,0} = \Gamma(\gamma) \int_{0}^{x} \frac{1}{\beta_k} v_k^{\rho_k+\gamma} (x - \tilde{v}_{k+1})^{-\gamma} \frac{dv_{k+1}}{\tilde{v}_{k+1}}
   \int_{\Delta_k} \prod_{j=1}^{k-1} v_j^{\alpha_j' + \beta_j' \rho_k - 1} \prod_{j=k+1}^{n} v_j^{\alpha_j' + \beta_j' \rho_k - 1}
   (v_1 - 1 - \cdots - v_k - v_{k+2} - \cdots - v_n)^{-\rho_k - \gamma} dv_1 \wedge \cdots dv_k \wedge dv_{k+2} \wedge \cdots \wedge dv_n
\end{align}
(3.33)
The result is given as follows

\[
\hat{\xi}_{k,0} = \frac{1}{\beta_k} \frac{\pi}{\sin \pi(\rho_k + \gamma)} \sin \pi \gamma \frac{\pi}{\Gamma(\rho_k + 1) \Gamma(1 - \alpha'_{j} - \beta_{j}\rho_{k})} \prod_{j=1}^{k-1} \frac{\Gamma(1 - \alpha_j - \beta_j \rho_k)}{\Gamma(1 - \alpha_{j-1} - \beta_{j-1} \rho_{k})} \prod_{j=k+1}^{n} \frac{\Gamma(1 - \alpha_j - \beta_j \rho_k)}{\Gamma(1 - \alpha_{j-1} - \beta_{j-1} \rho_{k})}.
\]

(3.34)

When \( k = 1 \), we make a change of variables \((v_1, \ldots, v_n)\) into \((v_1 - 1 - v_2 - \cdots - v_n)/g(v), v_1 v_2, \cdots, v_1 v_n)\) and obtain a similar representation to (3.33) giving (3.34).

Hence we have the equality

\[
\hat{G}_k(x) = \hat{U}_k[G_k](x)
\]

where \( \hat{U}_k \) denotes

\[
\hat{U}_k(\rho_k, \{\alpha'_1 + \beta'_1 \rho_k\}, \{\alpha_j + \beta_j \rho_k\}_{j=1,j\neq k}) = \frac{1}{\beta_k} \frac{\pi}{\sin \pi \gamma} \frac{\pi}{\sin \pi(\rho_k + \gamma)} \frac{\pi}{\prod_{j=1,j\neq k}^{n} \sin \pi(\alpha_j + \beta_j \rho_k)} \prod_{j=1}^{n} \sin \pi \gamma \prod_{j=1}^{n} \sin \pi(\rho_k + \gamma)
\]

One can show that

(3.35)

\[
W(x) = U_0[F](x) + \sum_{k=1}^{n} U_k[G_k](x)
\]

where

(3.36)

\[
U_0 = \frac{\pi^n \sin \pi \alpha'_1}{\sin \pi \gamma \prod_{j=1}^{n} \sin \pi \alpha_j}
\]

\[
U_k = -\frac{\sin \pi(\rho_k + \gamma)}{\sin \pi \rho_k} \hat{U}_k
\]

Conjecture has thus been verified.
Fig. 2

Fig. 3

Fig. 4

Fig. 5
References


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