

## ON QUASI-HYPERGEOMETRIC FUNCTIONS

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*Dedicated to Richard Askey on the occasion of his 65th birthday*

ABSTRACT. We define quasi-hypergeometric functions of regular singular type and show that they are characterized by certain fractional differential equations on the one hand and by certain difference-differential equations on the other. Two examples of quasi-hypergeometric functions are given, namely quasi-algebraic functions and partition functions appearing in fractional exclusion statistics.

### 1. Introduction

L. Euler studied the Lambert series

$$F(x) = \sum_{n=0}^{\infty} \binom{\alpha + \beta n}{n} x^n,$$

which is intimately related to the transcendental equation

$$y - 1 = xy^\beta,$$

[4, 13]. Recently, this kind of function has been given considerable attention by physicists. They play an important part in conformal field theory and fractional exclusion statistics. There is a pioneering work by B. Sutherland connecting them with fractional exclusion statistics and Calogero-Sutherland models [14–16]. The second author has extended some of these results to fractional exclusion statistics of multispecies of particles [10, 11], which is based on the results in [8, 18]. This corresponds exactly to an extension of transcendental functions of the above type to multivariable ones.

In this note, we would like to generalize and give a mathematical background for these functions which we call “quasi-hypergeometric functions”. These functions appear as an extension of general hypergeometric functions. The latter satisfy a holonomic system of differential equations of Barnes-Mellin type by means of  $b$ -functions [1–3]. A modern observation also has been discussed in relation to toric analysis by Gelfand et al. [5, 6].

However, the quasi-hypergeometric functions  $F(x_1, \dots, x_n)$  which we define here do not satisfy differential equations. We first present the system of fractional differential equations with respect to  $x_1, \dots, x_n$  which  $F(x_1, \dots, x_n)$  satisfy. Next, we show that  $F(x_1, \dots, x_n)$  also satisfies a kind of difference-differential equations with respect to  $x_1, \dots, x_n$  and other extra parameters  $\alpha_1, \dots, \alpha_r; \alpha'_1, \dots, \alpha'_s$  (analog of contiguous relations for hypergeometric functions).

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We can characterize these functions as the unique solutions to these functional equations.

## 2. System of fractional differential equations

Let  $\alpha, \beta \in \mathbf{C}$ ,  $\sigma = (\sigma_1, \dots, \sigma_n)$ ,  $\sigma_j > 0$  be given. We define a fractional derivative operator of order  $-\beta$ ,

$$P_\sigma(\alpha, \beta)f(x) = \frac{1}{\Gamma(\beta)} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} f(t^{\sigma_1} x_1, \dots, t^{\sigma_n} x_n) dt$$

for a smooth function in a neighbourhood  $\mathcal{U}$  of the origin of  $\mathbf{C}^n$  (see [17]).

We assume that  $\mathcal{U}$  is a Reinhardt domain, i.e.,  $x = (x_1, \dots, x_n) \in \mathcal{U}$  implies that  $(\rho_1 x_1, \dots, \rho_n x_n) \in \mathcal{U}$  for arbitrary complex numbers  $\rho_j$ , such that  $|\rho_j| \leq 1$ .

If  $\alpha$  and  $\beta$  are positive,  $P_\sigma(\alpha, \beta)$  is a well-defined operator, otherwise we may define it as a finite part of integrals at  $t = 0$  or  $t = 1$  in the sense of Hadamard [7].

The operator  $P_\sigma(\alpha, \beta)$  is an operator reminiscent of the one which Humbert and Agarwal [9] defined for the function of Mittag-Leffler:

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(1+n\beta)} x^n.$$

$P_\sigma(\alpha, \beta)$  satisfies the following basic properties.

**Proposition 1.** (i) For two arbitrary triples  $(\alpha, \beta, \sigma)$  and  $(\alpha', \beta', \sigma')$ ,  $P_\sigma(\alpha, \beta)$   $P_{\sigma'}(\alpha', \beta')$  commute with each other, i.e.,

$$P_\sigma(\alpha, \beta) \cdot P_{\sigma'}(\alpha', \beta') = P_{\sigma'}(\alpha', \beta') \cdot P_\sigma(\alpha, \beta).$$

(ii)  $P_\sigma(\alpha, 0)$  is the identity operator.

Furthermore, if  $\beta$  is a negative integer, say  $\beta = -m$ ,  $m = 1, 2, 3, \dots$ , then  $P_\sigma(\alpha, -m)$  reduces to a differential operator of order  $m$ ,

$$P_\sigma(\alpha, -m)f(x) = \prod_{k=1}^m \left( \alpha - k + \sum_{j=1}^n \sigma_j x_j \frac{\partial}{\partial x_j} \right) f(x).$$

For example,

$$\begin{aligned} P_\sigma(\alpha, -1)f(x) &= \left( \alpha - 1 + \sum_{j=1}^n \sigma_j x_j \frac{\partial}{\partial x_j} \right) f(x), \\ P_\sigma(\alpha, -2)f(x) &= \left[ (\alpha - 1)(\alpha - 2) + \sum_{j=1}^n (2(\alpha - 1)\sigma_j + \sigma_j^2) x_j \frac{\partial}{\partial x_j} \right. \\ &\quad \left. + \sum_{j,k=1}^n \sigma_j \sigma_k x_j x_k \frac{\partial^2}{\partial x_j \partial x_k} \right] f(x). \end{aligned}$$

(iii)

$$P_\sigma(\alpha + \beta, -\beta) \cdot P_\sigma(\alpha, \beta) = P_\sigma(\alpha, \beta) \cdot P_\sigma(\alpha + \beta, -\beta) = 1$$

so that  $P_\sigma(\alpha + \beta, -\beta)$  can be regarded as the inverse of  $P_\sigma(\alpha, \beta)$ .

(iv) For a monomial  $x_1^{\nu_1} \cdots x_n^{\nu_n}$ , we have

$$P_\sigma(\alpha, \beta)(x_1^{\nu_1} \cdots x_n^{\nu_n} f(x)) = x_1^{\nu_1} \cdots x_n^{\nu_n} P_\sigma\left(\alpha + \sum_{j=1}^n \sigma_j \nu_j, \beta\right) f(x).$$

(v)

$$\frac{\partial}{\partial x_k} P_\sigma(\alpha, \beta) f(x) = P_\sigma(\alpha + \sigma_k, \beta) \frac{\partial}{\partial x_k} f(x).$$

(vi)

$$P_\sigma(\alpha, \beta) x_1^{\nu_1} \cdots x_n^{\nu_n} = \frac{\Gamma(\alpha + \sum_{j=1}^n \sigma_j \nu_j)}{\Gamma(\alpha + \beta + \sum_{j=1}^n \sigma_j \nu_j)} x_1^{\nu_1} \cdots x_n^{\nu_n}.$$

The proof of Proposition 1 is almost immediate except for (iii), which follows from the following lemma.

**Lemma 1.**

$$P_\sigma(\alpha, \beta) \cdot P_\sigma(\alpha', \beta') f(x) = \int_0^1 g(t) f(t^{\sigma_1} x_1, \dots, t^{\sigma_n} x_n) \frac{dt}{t}$$

where  $g(s)$  denotes the Gauss hypergeometric function,

$$g(s) = \frac{1}{\Gamma(\beta + \beta')} (1-s)^{\beta + \beta' - 1} s^{\alpha - \beta'} F\left(\beta', \alpha' + \beta' - \alpha, \beta + \beta' \mid \frac{s-1}{s}\right).$$

In particular, the following co-cycle property holds:

$$\begin{aligned} P_\sigma(\alpha + \beta', \beta) \cdot P_\sigma(\alpha, \beta') &= P_\sigma(\alpha, \beta') \cdot P_\sigma(\alpha + \beta', \beta) \\ &= P_\sigma(\alpha, \beta + \beta'), \end{aligned} \quad (1)$$

which implies (iii) in Proposition 1 if  $\beta + \beta' = 0$ .

We now define a system of fractional differential equations (E) as follows.

Let  $\alpha_1, \dots, \alpha_r, \alpha'_1, \dots, \alpha'_s$  be  $r + s$  complex numbers, and  $\beta_i = (\beta_{ij})_{j=1}^n \in \mathbf{R}_+^n$  ( $1 \leq i \leq r$ ),  $\beta'_i = (\beta'_{ij})_{j=1}^n \in \mathbf{R}_+^n$  ( $1 \leq i \leq s$ ) be  $r + s$  tuples of  $n$ -dimensional vectors with non-negative components.

We assume that the following relations hold. For each  $j$ ,

$$\sum_{i=1}^s \beta'_{ij} = \sum_{i=1}^r \beta_{ij} + 1. \quad (2)$$

This condition assures that the function  $F(x)$  has *tempered growth* along a radial direction at the singularities, i.e., it has only a *regular singularity*.

We consider the following system of fractional differential equations for a function  $F = F(x)$  depending on the variables  $x_1, \dots, x_n$ :

$$(E) \quad \frac{\partial}{\partial x_j} F = \prod_{i=1}^s P_{\beta'_i}(\alpha'_i + \beta'_{ij}, -\beta'_{ij}) \cdot \prod_{i=1}^r P_{\beta_i}(\alpha_i, \beta_{ij}) F \quad (3)$$

for  $1 \leq j \leq n$ .

As is seen from (1), this system satisfies the compatibility condition

$$\begin{aligned}
\frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_k} F \right) &= \frac{\partial}{\partial x_j} \prod_{i=1}^r P_{\beta_i}(\alpha_i, \beta_{ik}) \prod_{i=1}^s P_{\beta'_i}(\alpha'_i + \beta'_{ik}, -\beta'_{ik}) F \\
&= \prod_{i=1}^r P_{\beta_i}(\alpha_i + \beta_{ij}, \beta_{ik}) \prod_{i=1}^s P_{\beta'_i}(\alpha'_i + \beta'_{ik} + \beta'_{ij}, -\beta'_{ik}) \\
&\quad \times \prod_{i=1}^r P_{\beta_i}(\alpha_i, \beta_{ij}) \prod_{i=1}^s P_{\beta'_i}(\alpha'_i + \beta'_{ij}, -\beta'_{ij}) F \\
&= \prod_{i=1}^r P_{\beta_i}(\alpha_i, \beta_{ij} + \beta_{ik}) \cdot P_{\beta'_i}(\alpha'_i + \beta'_{ij} + \beta'_{ik}, -\beta'_{ij} - \beta'_{ik}) F \\
&= \frac{\partial}{\partial x_k} \left( \frac{\partial}{\partial x_j} F \right)
\end{aligned}$$

because of symmetry.

### 3. Quasi-hypergeometric functions

By using the parameters in the preceding section, we consider the following power series in  $x$  at the origin.

$$\begin{aligned}
F \left( \begin{matrix} \{\alpha'_1; \beta'_1\}, \dots, \{\alpha'_s; \beta'_s\} \\ \{\alpha_1; \beta_1\}, \dots, \{\alpha_r; \beta_r\} \end{matrix} \middle| x \right) \\
= \sum_{\nu_1, \dots, \nu_n \geq 0} \frac{\prod_{i=1}^s \Gamma(\alpha'_i + \sum_{j=1}^n \beta'_{ij} \nu_j)}{\prod_{i=1}^r \Gamma(\alpha_i + \sum_{j=1}^n \beta_{ij} \nu_j) \nu_1! \cdots \nu_n!} x_1^{\nu_1} \cdots x_n^{\nu_n}. \quad (4)
\end{aligned}$$

We first remark that the following lemma holds by Stirling's formula:

**Lemma 2.** *We fix  $a, b \in \mathbf{R}$ ,  $k = 1, 2, 3, \dots$ . Then, for a large positive number  $t$ , there exists a positive constant  $C_0$  such that*

$$\frac{\Gamma(a+t)}{\Gamma(\frac{b+t}{k})^k} \leq C_0 t^{a-b+\frac{1}{2}(k-1)} k^t.$$

As a consequence of this lemma, we have

**Lemma 3.** *There exists a positive constant  $C_1$  such that*

$$\begin{aligned}
\left| \frac{\prod_{i=1}^s \Gamma(\alpha'_i + \sum_{j=1}^n \beta'_{ij} \nu_j)}{\prod_{i=1}^r \Gamma(\alpha_i + \sum_{j=1}^n \beta_{ij} \nu_j) \nu_1! \cdots \nu_n!} \right| \\
\leq C_1 \left( \sum_{j=1}^n b'_j \nu_j \right)^{\alpha'_{1,2,\dots,s} - \alpha_{1,2,\dots,r} + \frac{1}{2}(-n+s-1)} \cdot (r+n)^{b'_1 \nu_1 + \cdots + b'_n \nu_n}
\end{aligned}$$

where  $\alpha_{1,2,\dots,r}$ ,  $\alpha'_{1,2,\dots,s}$  and  $b_j$ ,  $b'_j$  denote the sums  $\alpha_1 + \cdots + \alpha_r$ ,  $\alpha'_1 + \cdots + \alpha'_s$ ,  $\sum_{i=1}^r \beta_{ij}$  and  $\sum_{i=1}^s \beta'_{ij}$ , respectively, such that  $b'_j = b_j + 1$ .

*Proof.* We assume that  $\nu_1, \dots, \nu_n$  are so large that  $\alpha'_i + \sum_{j=1}^n \beta'_{ij} \nu_j > 1$ .

We first note the inequality

$$\begin{aligned} & \left| \frac{\prod_{i=1}^s \Gamma(\alpha'_i + \sum_{j=1}^n \beta'_{ij} \nu_j)}{\prod_{i=1}^r \Gamma(\alpha'_{1,2,\dots,s} + \sum_{j=1}^n b'_j \nu_j)} \right| \\ &= \int_{1 \geq t_1 + \dots + t_{s-1}, t_j \geq 0} t_1^{\alpha'_1 + \sum_{j=1}^n \beta'_{1j} \nu_j - 1} \dots t_{s-1}^{\alpha'_{s-1} + \sum_{j=1}^n \beta'_{s-1,j} \nu_j - 1} \\ & \quad \times (1 - t_1 - \dots - t_{s-1})^{\alpha'_s + \sum_{j=1}^n \beta'_{sj} \nu_j - 1} dt_1 \wedge \dots \wedge dt_{s-1} \\ & \leq \frac{1}{(s-1)!} \end{aligned}$$

since the integrand on the right-hand side is smaller than 1.  $\square$

On the other hand, by the log convexity of the Gamma function  $\Gamma(x)$  for  $x > 0$ , we have

$$\prod_{i=1}^r \Gamma\left(\alpha_i + \sum_{j=1}^n \beta_{ij} \nu_j\right) \nu_1! \dots \nu_n! \geq \Gamma\left(\frac{\alpha_{1,2,\dots,r} + n + \sum_{i=1}^r \sum_{j=1}^n b'_j \nu_j}{n+r}\right)^{n+r}.$$

These two inequalities imply Lemma 3 from Lemma 2.

As a consequence of Lemma 3, the series (4) converges in the polydisc  $D$  defined by

$$|x_1| < (r+n)^{-b'_1}, \dots, |x_n| < (r+n)^{-b'_n},$$

so that the function (4) defines a holomorphic function at the origin.

Furthermore, we have

**Theorem 1.** *The function  $F$  satisfies the equations (E) and can be characterized as the unique solution to (E) which is holomorphic at the origin and*

$$F(0) = \frac{\prod_{i=1}^s \Gamma(\alpha'_i)}{\prod_{i=1}^r \Gamma(\alpha_i)}. \quad (5)$$

*Proof.* Assume that the holomorphic function at the origin

$$F(x) = \sum_{\nu_1 \geq 0, \dots, \nu_n \geq 0} a_{\nu_1, \dots, \nu_n} x_1^{\nu_1} \dots x_n^{\nu_n} \quad (6)$$

satisfies the equations (E). We fix  $j$ . From (vi) in Proposition 1, we have

$$P_{\beta'_i}(\alpha'_i + \beta'_{ij}, -\beta'_{ij}) x_1^{\nu_1} \dots x_n^{\nu_n} = \frac{\Gamma(\alpha'_i + \beta'_{ij} + \sum_{k=1}^n \beta'_{ik} \nu_k)}{\Gamma(\alpha'_i + \sum_{k=1}^n \beta'_{ik} \nu_k)} x_1^{\nu_1} \dots x_n^{\nu_n}$$

and

$$P_{\beta_i}(\alpha_i, \beta_{ij}) x_1^{\nu_1} \dots x_n^{\nu_n} = \frac{\Gamma(\alpha_i + \sum_{k=1}^n \beta_{ik} \nu_k)}{\Gamma(\alpha_i + \beta_{ij} + \sum_{k=1}^n \beta_{ik} \nu_k)} x_1^{\nu_1} \dots x_n^{\nu_n}.$$

The equations (E) give the recurrence relations with respect to  $\nu_1, \nu_2, \dots, \nu_n$  as

$$\begin{aligned} (\nu_j + 1) a_{\nu_1, \dots, \nu_j+1, \dots, \nu_n} &= \prod_{i=1}^r \frac{\Gamma(\alpha_i + \sum_{k=1}^n \beta_{ik} \nu_k)}{\Gamma(\alpha_i + \beta_{ij} + \sum_{k=1}^n \beta_{ik} \nu_k)} \\ & \quad \times \prod_{i=1}^s \frac{\Gamma(\alpha'_i + \beta'_{ij} + \sum_{k=1}^n \beta'_{ik} \nu_k)}{\Gamma(\alpha'_i + \sum_{k=1}^n \beta'_{ik} \nu_k)} \cdot a_{\nu_1, \dots, \nu_j, \dots, \nu_n}. \end{aligned}$$

These relations determine uniquely the coefficients  $a_{\nu_1, \dots, \nu_j, \dots, \nu_n}$  except for a constant factor. If  $a_{0, \dots, 0}$  equals (5), then  $F(x)$  coincides with

$$F\left(\begin{array}{c} \{\alpha'_1; \beta'_1\}, \dots, \{\alpha'_s; \beta'_s\} \\ \{\alpha_1; \beta_1\}, \dots, \{\alpha_r; \beta_r\} \end{array} \middle| x\right).$$

Thus the theorem has been proved.  $\square$

As a function of  $\alpha_1, \dots, \alpha_r, \alpha'_1, \dots, \alpha'_s$  and  $x$ , the function

$$F\left(\begin{array}{c} \{\alpha'_1; \beta'_1\}, \dots, \{\alpha'_s; \beta'_s\} \\ \{\alpha_1; \beta_1\}, \dots, \{\alpha_r; \beta_r\} \end{array} \middle| x\right)$$

is meromorphic in  $\alpha_1, \dots, \alpha_r, \alpha'_1, \dots, \alpha'_s$  in  $\mathbf{C}^{r+s}$  and holomorphic in  $x$  in the polydisc  $D$ .

When  $\beta_{ij}$  and  $\beta'_{ij}$  are integers, the functions (4) are nothing more than general hypergeometric functions of Barnes-Mellin type.

#### 4. System of difference-differential equations

In the preceding section, we have assumed that the parameters  $\beta_{ij}$  are positive. This restriction is sometimes too restrictive.

*In this section, we do not impose this condition on  $\beta_{ij}$ .*

We consider a function  $F = F(x; \alpha; \alpha')$  depending on the  $(n + r + s)$  variables,  $x = (x_1, \dots, x_n)$ ,  $\alpha = (\alpha_1, \dots, \alpha_r)$ , and  $\alpha' = (\alpha'_1, \dots, \alpha'_s)$ .

We denote by  $T_{\alpha_i}, T_{\alpha'_i}$  the shift operators deriving from the displacements  $\alpha_i \rightarrow \alpha_i + 1, \alpha'_i \rightarrow \alpha'_i + 1$ ,

$$\begin{aligned} T_{\alpha_i} f(x; \alpha_1, \dots, \alpha_i, \dots, \alpha_r; \alpha') &= f(x; \alpha_1, \dots, \alpha_i + 1, \dots, \alpha_r; \alpha'), \\ T_{\alpha'_i} f(x; \alpha_1, \dots, \alpha_i, \dots, \alpha_r; \alpha') &= f(x; \alpha; \alpha'_1, \dots, \alpha'_i + 1, \dots, \alpha'_s), \end{aligned}$$

and also by  $T_{\alpha_i}^a, T_{\alpha'_i}^a$  the shift operators of the displacements  $\alpha_i \rightarrow \alpha_i + a, \alpha'_i \rightarrow \alpha'_i + a$ , respectively.

We consider the system of difference-differential equations (E\*):

$$(E^*) \quad \left\{ \begin{array}{l} F = \left( \alpha_i + \sum_{k=1}^n \beta_{ik} x_k \frac{\partial}{\partial x_k} \right) T_{\alpha_i} F, \quad 1 \leq i \leq r, \quad (7) \\ T_{\alpha'_i} F = \left( \alpha'_i + \sum_{k=1}^n \beta'_{ik} x_k \frac{\partial}{\partial x_k} \right) F, \quad 1 \leq i \leq s, \quad (8) \\ \frac{\partial}{\partial x_j} F = T_{\alpha_1}^{\beta_{1j}} \dots T_{\alpha_r}^{\beta_{rj}} \cdot T_{\alpha'_1}^{\beta'_{1j}} \dots T_{\alpha'_s}^{\beta'_{sj}} F, \quad 1 \leq j \leq n. \quad (9) \end{array} \right.$$

Then we have the following theorem.

**Theorem 2.** *The function (4) satisfies the equations (E\*). It is characterized as the unique solution to (E\*) which satisfies the initial condition*

$$F(0; \alpha; \alpha') = \frac{\prod_{i=1}^s \Gamma(\alpha'_i)}{\prod_{i=1}^r \Gamma(\alpha_i)}.$$

*Proof.* The function (4) satisfies (7) and (8) because of the equalities

$$\begin{aligned} T_{\alpha_i} \Gamma \left( \alpha_i + \sum_{k=1}^n \beta_{ik} \nu_k \right) &= \left( \alpha_i + \sum_{k=1}^n \beta_{ik} \nu_k \right) \Gamma \left( \alpha_i + \sum_{k=1}^n \beta_{ik} \nu_k \right), \\ T_{\alpha'_i} \Gamma \left( \alpha'_i + \sum_{k=1}^n \beta'_{ik} \nu_k \right) &= \left( \alpha'_i + \sum_{k=1}^n \beta'_{ik} \nu_k \right) \Gamma \left( \alpha'_i + \sum_{k=1}^n \beta'_{ik} \nu_k \right). \end{aligned}$$

As for (9), we have

$$\begin{aligned} \frac{\partial}{\partial x_j} F(x) &= \sum_{\nu_1, \dots, \nu_n \geq 0} \frac{\prod_{i=1}^s \Gamma \left( \alpha'_i + \beta'_{ij} + \sum_{k=1}^n \beta'_{ik} \nu_k \right)}{\prod_{i=1}^r \Gamma \left( \alpha_i + \beta_{ij} + \sum_{k=1}^n \beta_{ik} \nu_k \right) \nu_1! \cdots \nu_n!} x_1^{\nu_1} \cdots x_n^{\nu_n} \\ &= \text{the right-hand side of (9)}. \end{aligned}$$

Conversely, assume that  $F(x)$  has the expansion (6) at the origin  $x = 0$  such that  $a_{\nu_1, \dots, \nu_n} = a_{\nu_1, \dots, \nu_n}(\alpha, \alpha')$  depend on  $\alpha, \alpha'$  meromorphically. From (9), we have the recurrence relations

$$\begin{aligned} (\nu_j + 1) a_{\nu_1, \dots, \nu_j+1, \dots, \nu_n}(\alpha, \alpha') \\ = a_{\nu_1, \dots, \nu_j, \dots, \nu_n}(\alpha_1 + \beta_{1j}, \dots, \alpha_r + \beta_{rj}; \alpha'_1 + \beta'_{1j}, \dots, \alpha'_s + \beta'_{sj}), \end{aligned}$$

so that  $a_{\nu_1, \dots, \nu_j, \dots, \nu_n}(\alpha, \alpha')$  are uniquely determined from  $a_{0, \dots, 0}(\alpha, \alpha')$ .

The last one satisfies the difference equations from (7) and (8):

$$T_{\alpha_i} a_{0, \dots, 0}(\alpha, \alpha') = \alpha_i^{-1} a_{0, \dots, 0}(\alpha, \alpha'), \quad T_{\alpha'_i} a_{0, \dots, 0}(\alpha, \alpha') = \alpha'_i a_{0, \dots, 0}(\alpha, \alpha').$$

A general solution to these can be expressed as

$$a_{0, \dots, 0}(\alpha, \alpha') = \frac{\prod_{i=1}^s \Gamma(\alpha'_i)}{\prod_{i=1}^r \Gamma(\alpha_i)} \cdot H(\alpha, \alpha') \quad (10)$$

where  $H(\alpha, \alpha')$  denotes an arbitrary periodic function with the periods 1 relative to each variable  $\alpha_i, \alpha'_i$ .

In particular, if one takes  $H(\alpha, \alpha') = 1$ ,  $F(x)$  coincides with

$$F \left( \begin{array}{c} \{\alpha'_1; \beta'_1\}, \dots, \{\alpha'_s; \beta'_s\} \\ \{\alpha_1; \beta_1\}, \dots, \{\alpha_r; \beta_r\} \end{array} \middle| x \right). \quad \square$$

We now fix a system of integers  $\mathbf{l} = (l_1, \dots, l_r)$  and  $\mathbf{l}' = (l'_1, \dots, l'_s)$ . We can take as  $H(\alpha, \alpha')$  the periodic function

$$H(\alpha, \alpha') = \exp \left[ 2\pi i \left( \sum_{\mu=1}^r l_\mu \alpha_\mu + \sum_{\mu=1}^s l'_\mu \alpha'_\mu \right) \right], \quad (11)$$

then we have the solution  $F(x)$  to (E\*) which has the expression

$$F(x) = \exp \left[ 2\pi i \left( \sum_{\mu=1}^r l_\mu \alpha_\mu + \sum_{\mu=1}^s l'_\mu \alpha'_\mu \right) \right] F \left( \begin{array}{c} \{\alpha'_1; \beta'_1\}, \dots, \{\alpha'_s; \beta'_s\} \\ \{\alpha_1; \beta_1\}, \dots, \{\alpha_r; \beta_r\} \end{array} \middle| x^* \right)$$

where  $x^* = (x_1^*, \dots, x_n^*)$  denotes the point such that

$$x_1^* = x_1 \exp \left[ 2\pi i \left( \sum_{\mu=1}^r l_\mu \beta_{\mu 1} + \sum_{\mu=1}^s l'_\mu \beta'_{\mu 1} \right) \right], \dots, x_n^* = x_n \exp \left[ 2\pi i \left( \sum_{\mu=1}^r l_\mu \beta_{\mu n} + \sum_{\mu=1}^s l'_\mu \beta'_{\mu n} \right) \right].$$

We shall abbreviate these solutions as  $F_{\mathbb{I}'}(x)$ ,

Since an arbitrary periodic function  $H(\alpha, \alpha')$  has a Fourier expansion by using a sequence (11), we can conclude the following.

**Proposition 2.** *Every solution to (E\*) is a linear combination of a countable number of the solutions  $F_{\mathbb{I}'}(x)$ .*

## 5. Examples

**Example 1.** Let  $\alpha_1, \alpha_2 \in \mathbf{R}$  and  $\beta_1, \beta_2 \in \mathbf{R}_+$  be given such that  $\beta_1 + \beta_2 = 1$ . The function

$$F = F(\{\alpha_1; \beta_1\}, \{\alpha_2; \beta_2\} | x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \beta_1 n) \Gamma(\alpha_2 + \beta_2 n)}{n!} x^n$$

converges in the disc  $|x| < \beta_1^{-\beta_1} \beta_2^{-\beta_2}$  and satisfies the equation (E):

$$\frac{dF}{dx} = P_{\beta_1}(\alpha_1 + \beta_1, -\beta_1) P_{\beta_2}(\alpha_2 + \beta_2, -\beta_2) F. \quad (12)$$

The function  $F(\{\alpha_1; \beta_1\}, \{\alpha_2; \beta_2\} | x)$  is the unique solution to (E) which is holomorphic at the origin and such that  $F(0) = \Gamma(\alpha_1) \Gamma(\alpha_2)$ . It also satisfies the equation (E\*):

$$T_{\alpha_1} F = (\alpha_1 + \beta_1 x \frac{d}{dx}) F, \quad T_{\alpha_2} F = (\alpha_2 + \beta_2 x \frac{d}{dx}) F, \quad \frac{d}{dx} F = T_{\alpha_1}^{\beta_1} T_{\alpha_2}^{\beta_2} F. \quad (13)$$

The equations (E\*) also are satisfied by the functions

$$F_{l_1, l_2}(x) = \exp[2\pi i(l_1 \alpha_1 + l_2 \alpha_2)] F(\{\alpha_1; \beta_1\}, \{\alpha_2; \beta_2\} | \exp[2\pi i(l_1 \beta_1 + l_2 \beta_2)] x)$$

for all  $(l_1, l_2) \in \mathbf{Z}^2$ .

$F_{l_1, l_2}$  does not satisfy the equation (E) but instead satisfies

$$\frac{dF}{dx} = \exp[2\pi i(l_1 \beta_1 + l_2 \beta_2)] P_{\beta_1}(\alpha_1 + \beta_1, -\beta_1) P_{\beta_2}(\alpha_2 + \beta_2, -\beta_2) F. \quad (14)$$

It is characterized as the unique solution to (13), which is holomorphic at the origin and  $F(0) = \exp[2\pi i(l_1 \alpha_1 + l_2 \alpha_2)] \Gamma(\alpha_1) \Gamma(\alpha_2)$ .

On the other hand, by using the equalities

$$\frac{\Gamma(\alpha_1 + \beta_1 n) \Gamma(\alpha_2 + \beta_2 n)}{\Gamma(\alpha_1 + \alpha_2 + n)} = \int_0^{\infty} u^{\alpha_1 + \beta_1 n - 1} (1 + u)^{-\alpha_1 - \alpha_2 - n} du$$

and the binomial expansion

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \alpha_2 + n)}{n!} x^n = \Gamma(\alpha_1 + \alpha_2) (1 - x)^{-\alpha_1 - \alpha_2},$$

we get the integral expression for  $F(x)$  given by

$$F(x) = \Gamma(\alpha_1 + \alpha_2) \int_0^{\infty} u^{\alpha_1 - 1} (1 + u - u^{\beta_1} x)^{-\alpha_1 - \alpha_2} du \quad (15)$$

for  $|x| < \beta_1^{-\beta_1} \beta_2^{-\beta_2}$ . We simply denote the number  $\beta_1^{-\beta_1} \beta_2^{-\beta_2}$  by  $c$ .

At  $x = 0$ , the quasi-algebraic equation

$$1 + u - xu^{\beta_1} = 0$$

has the two particular solutions  $u = u_+(x)$ ,  $u_-(x)$

$$u_+(x) = -1 + e^{\pi i \beta_1} x + \cdots, \quad u_-(x) = -1 + e^{-\pi i \beta_1} x + \cdots,$$



whose coefficients are complex conjugates of each other. When  $x$  increases and approaches  $c$ , then  $u_{\pm}(x)$  move in the upper (lower) half plane and approach the positive number  $\beta_1/\beta_2$ . One sees that the function (15) has a singularity of the braid type at the point  $x = c$ .

**Example 2.** Let  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{R}$  be given such that  $\beta_1 = \beta_2 + 1$ . Consider the function

$$F = F\left(\begin{matrix} \{\alpha_1; \beta_1\} \\ \{\alpha_2; \beta_2\} \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \beta_1 n)}{\Gamma(\alpha_2 + \beta_2 n) n!} x^n$$

in the following two cases.

Case (i)  $\beta_1 > 1, \beta_2 > 0$ .  $F$  converges for  $|x| < c$ ,  $c = \beta_1^{-\beta_1}(\beta_1 - 1)^{\beta_1 - 1}$ , and is the unique solution to

$$(E) \quad \frac{dF}{dx} = P_{\beta_1}(\alpha_1 + \beta_1, -\beta_1) P_{\beta_2}(\alpha_2, \beta_2) F,$$

such that  $F(0) = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_2)}$ .  $F$  also satisfies the equations (E\*):

$$T_{\alpha_1} F = (\alpha_1 + \beta_1 x \frac{d}{dx}) F, \quad F = (\alpha_2 + \beta_2 x \frac{d}{dx}) T_{\alpha_2} F, \quad \frac{d}{dx} F = T_{\alpha_1}^{\beta_1} T_{\alpha_2}^{\beta_2} F. \quad (16)$$

$F$  has the integral expression

$$F = -\frac{\Gamma(\alpha_1 + 1 - \alpha_2)}{2\pi i} \int_{\mathcal{L}} w^{\alpha_1 - 1} (1 - w + xw^{\beta_1})^{-\alpha_1 - 1 + \alpha_2} dw \quad (17)$$

for  $|x| < c$ . Assume for simplicity that  $0 < x < c$ . Then the path of integration  $\mathcal{L}$  is constructed as follows. There exist two positive solutions  $w_1, w_2$  to the equation

$$1 - w + xw^{\beta_1} = 0$$

such that  $1 < w_1 < w_2$ .

We construct a path  $\mathcal{L}$  starting from 0 in the lower half plane, crossing the interval  $[w_1, w_2]$  and going to 0 in the upper half plane.

When  $x$  tends to 0,  $w_1$  approaches 1, and the integral (17) is holomorphic in  $x$  at  $x = 0$ . On the other hand, when  $x$  approaches  $c$ , then  $w_1, w_2$  approach each other. Therefore, the integral (17) is no longer holomorphic at  $x = c$ . The function  $F(x)$  then has a singularity of braid type there. In particular, if  $\alpha_1 = \alpha_2$ , then  $F$  reduces to

$$F(x) = \sum_{n=0}^{\infty} \binom{\alpha_1 + \beta_1 n}{n} x^n = \frac{w_1^{\alpha_1}}{\beta_1 + (1 - \beta_1)w_1}$$

which is a well-known formula [13].

Case (ii)  $1 > \beta_1 > 0, 0 > \beta_2 > -1$ .

By using the Gauss identity  $\Gamma(\lambda)\Gamma(1 - \lambda) = \pi/\sin \pi\lambda$ ,  $F(x)$  can be written as

$$F(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \beta_1 n) \Gamma(1 - \alpha_2 - \beta_2 n) \sin \pi(\alpha_2 + \beta_2 n)}{n! \pi} x^n.$$

Hence,  $F(x)$  can be written as  $F = F_+ - F_-$  where

$$F_+(x) = \frac{\exp[\pi i \alpha_2]}{2\pi i} F(\{\alpha_1; \beta_1\}, \{1 - \alpha_2; -\beta_2\} \mid \exp[\pi i \beta_2] x),$$

$$F_-(x) = \frac{\exp[-\pi i \alpha_2]}{2\pi i} F(\{\alpha_1; \beta_1\}, \{1 - \alpha_2; -\beta_2\} \mid \exp[-\pi i \beta_2] x).$$

Each of them satisfies the equations of the type (14) which are different from each other. But both of them satisfy the same equations (E\*) and (16).

**Example 3.**

$$F = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^s \Gamma(\alpha'_k + \beta'_k n)}{\prod_{k=1}^r \Gamma(\alpha_k + \beta_k n) n!} x^n$$

for  $\beta_k, \beta'_k > 0$  and with the relation  $\beta_1 + \cdots + \beta_r + 1 = \beta'_1 + \cdots + \beta'_s$ , is a general one in the single variable case. In a way similar to Examples 1 and 2, one can show that the above series is convergent for  $|x| < c$  for  $c = \beta_1^{\beta_1} \cdots \beta_r^{\beta_r} \cdot \beta_1'^{-\beta_1} \cdots \beta_s'^{-\beta_s}$ .

$F$  satisfies the equations

$$(E) \quad \frac{d}{dx} F = \prod_{k=1}^r P_{\beta_k}(\alpha_k, \beta_k) \prod_{k=1}^s P_{\beta'_k}(\alpha'_k + \beta'_k, -\beta'_k) F,$$

$$(E^*) \quad \begin{cases} F = (\alpha_k + \beta_k x \frac{d}{dx}) T_{\alpha_k} F, & 1 \leq k \leq r, \\ T_{\alpha'_k} F = (\alpha'_k + \beta'_k x \frac{d}{dx}) F, & 1 \leq k \leq s, \\ \frac{d}{dx} F = T_{\alpha_1}^{\beta_1} \cdots T_{\alpha_r}^{\beta_r} T_{\alpha'_1}^{\beta'_1} \cdots T_{\alpha'_s}^{\beta'_s} F. \end{cases}$$

We will show in a subsequent article that  $F$  has a singularity at  $x = c$  and has a power series expansion near  $c$ , namely,

$$F(x) = (c - x)^\delta [a_0 + a_1(c - x) + a_2(c - x)^2 + \cdots] + (\text{a holomorphic function})$$

where  $\delta$  denotes  $\delta = \alpha_{1,2,\dots,r} - \alpha'_{1,2,\dots,s} + \frac{1}{2}(s - r - 1)$ .

In view of (iii) in Proposition 1, (E) is equivalent to

$$\prod_{k=1}^r P_{\beta_k}(\alpha_k + \beta_k, -\beta_k) \frac{d}{dx} F = \prod_{k=1}^s P_{\beta'_k}(\alpha'_k + \beta'_k, -\beta'_k) F.$$

When  $\beta_k = \beta'_k = 1$  for all  $k$ ,  $s$  must be equal to  $r + 1$ .  $F(x)$  reduces to the hypergeometric function of higher order [2, 3]

$${}_r F_{r+1} \left( \begin{matrix} \alpha'_1, \dots, \alpha'_{r+1} \\ \alpha_1, \dots, \alpha_r \end{matrix} \mid x \right).$$

(E) reduces to the ordinary differential equation

$$\prod_{k=1}^r (\alpha_k + x \frac{d}{dx}) \frac{d}{dx} F = \prod_{k=1}^{r+1} (\alpha'_k + x \frac{d}{dx}) F.$$

**Example 4.** We fix  $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbf{R}$ . The quasi-algebraic equation with respect to  $y$  given by

$$y^{\lambda_0} + x_1 y^{\lambda_1} + \cdots + x_n y^{\lambda_n} - 1 = 0$$

has a holomorphic solution in  $(x_1, \dots, x_n)$  at the origin such that  $y = 1$  for  $x = 0$ . Then, for an arbitrary  $\rho \in \mathbf{C}$ ,  $y^\rho$  has the expansion in  $x$ ,

$$y^\rho = \frac{\rho}{\lambda_0} \sum_{\nu_1 \geq 0, \dots, \nu_n \geq 0} (-1)^{|\nu|} \frac{\Gamma(A_\nu)}{\Gamma(A_\nu - |\nu| + 1) \nu_1! \cdots \nu_n!} x_1^{\nu_1} \cdots x_n^{\nu_n} \quad (18)$$

where  $A_\nu = \frac{1}{\lambda_0}(\rho + \lambda_1 \nu_1 + \cdots + \lambda_n \nu_n)$  and  $|\nu| = \nu_1 + \cdots + \nu_n$ , i.e.,

$$y^\rho = \frac{\rho}{\lambda_0} F \left( \begin{matrix} \{\frac{\rho}{\lambda_0}; (\frac{\lambda_1}{\lambda_0}, \dots, \frac{\lambda_n}{\lambda_0})\} \\ \{\frac{\rho}{\lambda_0} + 1; (\frac{\lambda_1}{\lambda_0} - 1, \dots, \frac{\lambda_n}{\lambda_0} - 1)\} \end{matrix} \middle| x_1, \dots, x_n \right).$$

In fact, assume that  $y^\rho$  has an expansion

$$y^\rho = 1 + \sum_{\nu_1 \geq 0, \dots, \nu_n \geq 0, |\nu| > 0} a_{\nu_1, \dots, \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}.$$

Then, by the Cauchy integral formula, we have for a small positive number  $\epsilon$ ,

$$a_{\nu_1, \dots, \nu_n} = \left( \frac{1}{2\pi i} \right)^n \int_{|x_1|=\epsilon, \dots, |x_n|=\epsilon} y^\rho x_1^{-\nu_1-1} \cdots x_n^{-\nu_n-1} dx_1 \wedge \cdots \wedge dx_n.$$

The change of variables of integration,  $(x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{n-1}, y)$ , gives

$$a_{\nu_1, \dots, \nu_n} = \left( \frac{1}{2\pi i} \right)^n \int_{|x_1|=\epsilon, \dots, |x_{n-1}|=\epsilon, |y-1|=\epsilon} y^\rho x_1^{-\nu_1-1} \cdots x_{n-1}^{-\nu_{n-1}-1} \times \frac{\partial x_n}{\partial y} dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy$$

for  $x_n = (1 - y^{\lambda_0} - x_1 y^{\lambda_1} - \cdots - x_{n-1} y^{\lambda_{n-1}}) / y^{\lambda_n}$ .

Using the binomial expansion  $y^\rho = \sum_{l=0}^{\infty} \binom{\rho}{l} (y-1)^l$ ,  $a_{\nu_1, \dots, \nu_n}$  is evaluated by residue calculus as in (18).

It has been known since H. Mellin that if we put  $\lambda_0 = n + 1$ ,  $\lambda_1 = n, \dots, \lambda_n = 1$ , then  $y$  reduces to a general algebraic function corresponding to the singularity of the A-type root system (see [3, 12], etc.).

**Example 5.** Consider the function

$$F(x) = \sum_{\nu_1, \dots, \nu_n \geq 0} \frac{\prod_{i=1}^n \Gamma(\alpha_i + \sum_{j=1}^n \beta'_{ij} \nu_j)}{\prod_{i=1}^n \Gamma(\alpha_i + \sum_{j=1}^n \beta_{ij} \nu_j) \nu_1! \cdots \nu_n!} x_1^{\nu_1} \cdots x_n^{\nu_n} \quad (19)$$

where we assume  $\beta'_{ij} = \beta_{ij} = -g_{ij}$  for  $i \neq j$  and  $\beta'_{ii} = \beta_{ii} + 1 = 1 - g_{ii}$  for suitable real numbers  $g_{ij}$ .

This function has been investigated in recent papers on statistical mechanics [10, 11] by the second author. It is described simply by using the function  $w_1^{\alpha_1} \cdots w_n^{\alpha_n}$  depending on  $x_1, \dots, x_n$ , which is derived from a system of the quasi-algebraic equations

$$w_i = 1 + x_i w_i^{1-g_{ii}} w_1^{-g_{i1}} \cdots w_n^{-g_{in}} \quad 1 \leq i \leq n. \quad (20)$$

These are the fundamental equations discovered by Wu [18] for describing mutual fractional exclusion statistics following Haldane [8] which is an extension of an earlier work by Sutherland [15] in the one variable case. The equations (20) can be solved explicitly as a power series expansion in  $x_1, \dots, x_n$  by the Lagrange inversion formula in the multivariable case, see [11] for details.

It seems an interesting problem to study the singularities and the monodromy property of  $F(x)$  when  $F(x)$  is analytically continued.

Recently Prof. V. S. Retakh pointed out to us that quasi-hypergeometric functions are very similar to the GG-functions defined by Gelfand and Graev [19]. They seem to obtain an equivalent form of our equations (E\*), although we have not yet shown this. They define GG-functions as a wider class of functions which are not necessarily of regular singular type. For geometric reasons, we only consider here quasi-hypergeometric functions of regular singular type.

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