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Generalized Lagrange theorem and thermodynamics of a multispecies quasiparticle gas with mutual fractional exclusion statistics

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We discuss the relationship between the classical Lagrange theorem in mathematics and the quantum statistical mechanics and thermodynamics of an ideal gas of multispecies quasiparticles with mutual fractional exclusion statistics. First, we show that the thermodynamic potential and the density of the system are analytically expressed in terms of the language of generalized cluster expansions, where the cluster coefficients are determined from Wu's functional relations for describing the distribution functions of mutual fractional exclusion statistics. Second, we generalize the classical Lagrange theorem for inverting the one complex variable functions to that for the multicomplex variable functions. Third, we explicitly obtain all the exact cluster coefficients by applying the generalized Lagrange theorem. [S0163-1829(98)03335-9]

I. INTRODUCTION

The concept of spin-charge separation[1] has attracted much interest from physicists in recent years. It has been believed that the concept is very significant and responsible for the origins of high T_c superconductivity,[2] quantum Hall effect,[3] anyon superconductivity,[4] and Mott transition.[5] The spin-charge separation is essentially realized by the two types of excitations—spin and charge excitations—in the system. A microscopic derivation for this was first carried out for the Tomonaga-Luttinger model (TLM),[6] the Calogero-Sutherland model (CSM),[7,8] and the Haldane-Shastry model (HSM).[9] The spin-charge separation resulted in the concept of Luttinger liquids in one dimension.[10] Since the spin and charge excitations are not independent of each other, this effect can be taken care of by mutual fractional exclusion statistics (MFES) (Refs. 11–15) between the two excitations. This situation was expected to be correct even for the higher-dimensional systems.[16] However, the system having the spin-charge separation is not the only system where MFES plays an important role; there are many other such systems. Therefore, it belongs to a broader category for the systems of multispecies quasiparticles with MFES. Thus, this class of systems seems to be very important in the study of strongly correlated systems. To understand the thermal properties of such a system of multispecies quasiparticles with MFES, it is inevitably necessary to consider quantum statistical mechanics (QSM) of the system in order to obtain the equation of state for the system. However, it has been extremely difficult to do so, except for the pure FES cases,[17–19] since MFES are given by a set of functional equations for Wu's distribution functions.[12]

Recently, some efforts have been made in this direction.[20–22] Isakov, Mashkevich, and Ouvry,[20] Mashkevich,[21] and Isakov and Mashkevich[22] have presented the analytic expressions of the cluster coefficients of cluster expansions for the systems with MFES. In order to obtain the expressions of cluster coefficients in the cluster expansion, they explicitly expanded Wu's functional relations up to several order terms of the expansion by using computer algebra, and for all the higher order coefficients the analytic expressions were conjectured. Although their results seem very plausible and reasonable, we have been unable to justify the analytic expressions of the cluster coefficients.

In this paper we would like to show that the classical Lagrange theorem and the Lagrange series expansion for the functions of one complex variable[23] can be generalized to those for the functions of many complex variables, and that these generalized results enable one to obtain all the exact cluster coefficients in the cluster expansion for a multispecies quasiparticle gas with MFES.[11–13]

The organization of the present paper is the following. In Sec. II, we introduce the QSM for a multispecies gas system. In Sec. III, we review Haldane's definition of FES and Wu's QSM formulation for the system of an ideal gas with MFES. In Sec. IV, we generalize the Sutherland transformation and we formulate the cluster expansions for a multispecies system with MFES. In Sec. V, we generalize the classical Lagrange theorem for inverting single complex variable functions to that for inverting multicomplex variable functions. This provides the generalized Lagrange expansions. In Sec. VI, we apply the generalized Lagrange expansions to obtain the exact coefficients of the generalized Lagrange series for the multispecies systems with MFES. In Sec. VII, we apply the coefficients of the generalized Lagrange series to represent the exact cluster coefficients for many physical systems. In Sec. VIII, we draw conclusions.

II. QUANTUM STATISTICAL MECHANICS

In QSM,[24] it is well known that the grand partition function Q for an M-species gas can be written as

$$Q=\sum_{N=0}^{\infty}\sum_{\Sigma_{a=1}^{M}N_a=N}Q_{N_1N_2\cdots N_M}z_1^{N_1}z_2^{N_2}\cdots z_M^{N_M}. \tag{1}$$

Here $Q_{N_1N_2\cdots N_M}$ $(=Q_N)$ means the microcanonical partition function for N particles and $N=\Sigma_{a=1}^{M}N_a$ such that $N_a=\Sigma_i N_i^a$, where N_a is the number of species a and N_i^a the number of species a of a good quantum number, i, respectively. $z_a = \exp[\beta\mu_a]$ stands for the fugacity of species a where μ_a is the chemical potential of species a and $\beta = 1/k_BT$. The thermodynamic potential Ω and the total number N of the system are given from Q as

$$\Omega = -PV = -k_BT \ln Q, \tag{2}$$

$$N = \sum_a N_a = \sum_a z_a \frac{\partial}{\partial z_a} \ln Q, \tag{3}$$

respectively. As we can expand Q as $Q = 1 + Q_{10\cdots 0}z_1 + \cdots$, we can expand $\ln Q$ as $(1/V) \ln Q = b_{10\cdots 0}z_1 + \cdots$. Hence, we can expand the Ω and N in the form of the cluster expansion[24] as

$$-\frac{\Omega}{Vk_BT} = \frac{P}{k_BT} = \sum_{L=1}^{\infty} \sum_{\Sigma_{a=1}^{M} l_a = L} b_{l_1 l_2 \cdots l_M} z_1^{l_1} z_2^{l_2} \cdots z_M^{l_M}, \tag{4a}$$

$$\frac{N}{V} = \sum_{L=1}^{\infty} \sum_{\Sigma_{a=1}^{M} l_a = L} L b_{l_1 l_2 \cdots l_M} z_1^{l_1} z_2^{l_2} \cdots z_M^{l_M}, \tag{4b}$$

where $b_{l_1 l_2 \cdots l_M}$ $(= b_L)$ are the cluster coefficients.[24]

For the one-species case of $M = 1$, one can write $P/k_BT = F(z) = \Sigma_{l=1}^{\infty} b_l z^l$ and $N/V = z(\partial/\partial z)F(z) = \Sigma_{l=1}^{\infty} l b_l z^l$. Then, Lee and Yang's theorem[25] tells us that *if there exists a singularity of $F(z)$ on the positive real axis of a complex z plane, then there is a phase transition in the system*. Therefore, the convergence of the cluster expansion is exactly related to the existence of a phase transition in the system. Thus, the explicit evaluation of the cluster coefficients is very important for the theory of phase transitions. However, since usually it is extremely difficult to analytically obtain the exact cluster coefficients for the multispecies cases,[20–22] much knowledge is still absent. The present paper will show that one can obtain the exact cluster coefficients for the system of a multispecies quasiparticle gas with MFES.[11–13]

III. MUTUAL FRACTIONAL EXCLUSION STATISTICS

A. Haldane's definition of MFES

Haldane's original definition of FES (Ref. 11) is generalized to the definition of MFES with a set of good quantum numbers so that the statistical interactions g_{ij}^{ab} are given by the differential relations

$$\Delta D_i^a = -\sum_{b,j} g_{ij}^{ab} \Delta N_j^b, \tag{5}$$

where D_i^a is the dimension of the Hilbert space H_i^a of states of a single particle of species a and good quantum number i, confined to a finite region of matter. D_i^a can change as particles are added, while keeping the boundary conditions and the size of the condensed-matter region constant. D_i^a is given by the integrated form

$$D_i^a = G_i^a - \sum_{j,b} g_{ij}^{ab}(N_j^b - \delta_{ij}\delta_{ab}), \tag{6}$$

where G_i^a is a constant and interpreted as the number of available single-particle states of species a with good quantum number i when no particle exists in the system.

B. Wu's formulation of QSM with MFES

Following the argument of Wu[12,16] for the QSM formulation of the MFES ideal gas, we define the microcanonical partition function Q_N by

$$Q_N = \sum_{N_i^a} W(\{N_i^a\}) e^{-\beta E(\{N_i^a\})}, \tag{7}$$

$$W(\{N_i^a\}) = \prod_{i,a} \frac{[D_i^a + N_i^a - 1]!}{N_i^a! [D_i^a - 1]!}, \tag{8}$$

where $E(\{N_i^a\}) = \Sigma_{i,a} \epsilon_i^a N_i^a$. The most probable distribution of Q is given by taking the extremum condition $\delta/\delta N_i^a \{\ln W(\{N_i^a\}) + \Sigma_{j,b} \beta(\epsilon_j^b - \mu_b) N_j^b\} = 0$. This yields Wu's distribution function w_i^a:

$$w_i^a = \frac{1}{n_i^a} - \sum_{j,b} g'^{ab}_{ij} \frac{n_j^b}{n_i^a}, \tag{9}$$

$$(1 + w_i^a) \prod_{j,b} \left(\frac{w_j^b}{1 + w_j^b} \right)^{g_{ji}^{ba}} = e^{\beta(\epsilon_i^a - \mu_a)}, \tag{10}$$

where $n_i^a \equiv N_i^a / G_i^a$ and $g'^{ab}_{ij} \equiv g_{ij}^{ab} G_j^b / G_i^a$. Now, the thermodynamic potential Ω and the total number N are written as

$$-\frac{\Omega}{Vk_BT} = \frac{P}{k_BT} = \frac{1}{V} \sum_{i,a} G_i^a \ln \left(\frac{1 + w_i^a}{w_i^a} \right), \tag{11}$$

$$\frac{N}{V} = \frac{1}{V} \sum_{i,a} G_i^a n_i^a = \sum_{i,a} z_i^a \frac{\partial}{\partial z_i^a} \left(\frac{P}{k_BT} \right), \tag{12}$$

respectively. They are valid for all the cases with the different species and a set of good quantum numbers.

Here, we would like to note that Isakov[13] independently obtained results similar to Wu's at almost the same time Wu published his results (on the Internet) (however, Wu first obtained his results about two years before publication).

IV. SUTHERLAND TRANSFORMATIONS AND CLUSTER EXPANSIONS

The main difficulties for calculating the thermodynamic potential lie in the fact that Eq. (10) represents a set of very complicated functional equations for MFES. Therefore, when MFES is taken into account, there has been no good method to explicitly obtain the cluster coefficients in the cluster expansion from Wu's distribution functions w_i^a, although several authors[20–22] have attempted to obtain them by considering the special cases of multispecies systems with MFES, explicitly expanding Wu's functional relations. We now present a method to resolve this problem.

A. Sutherland transformations

Let us define the following transformation:

$$\zeta_i^a = \frac{1+w_i^a}{w_i^a}. \tag{13}$$

This is a generalization of the transformation that is first adopted by Sutherland[8] for the Calogero-Sutherland model (CSM),[7,8] and hence $w_i^a = 1/(\zeta_i^a - 1)$. Substituting Eq. (13) into Eqs. (10)–(12), we obtain the following:

$$[(\zeta_i^a)^{g_{ii}^{aa}} - (\zeta_i^a)^{g_{ii}^{aa}-1}] \prod_{j,b(\neq i,a)} (\zeta_j^b)^{g_{ji}^{ba}} = e^{\beta(\mu_a - \epsilon_i^a)} \equiv \alpha_i^a, \tag{14}$$

$$\frac{P}{k_B T} = \frac{1}{V} \sum_{i,a} G_i^a \ln \zeta_i^a, \tag{15}$$

$$\frac{N}{V} = \frac{1}{V} \sum_{i,a} G_i^a z_i^a \frac{\partial}{\partial z_i^a} \ln \zeta_i^a. \tag{16}$$

B. Cluster expansions for the multispecies systems with MFES

The problem now is how one can obtain the power series expansion of $\ln \zeta_i^a$ with respect to the fugacity. Let us denote as $\ln \zeta_i^a \equiv \omega_i^a$ and suppose that ω_i^a is expanded in the following form:

$$\omega_i^a \equiv {\sum_{\{l_i^a\}=0}^{\infty}}' \; c_i^a[\{l_i^a\}] \prod_{j,b} (\alpha_j^b)^{l_j^b}, \tag{17}$$

where $c_i^a[\{l_i^a\}]$ stand for $c_{\{l_i^a\}i}{}^a(\{g_{ij}^{ab}\})$, which are functions of $\{l_i^a\}$ (the set of integers $\{l_1^a, l_2^b, \cdots\}$) and $\{g_{ij}^{ab}\}$ (the set of all MFES parameters), and $'$ means that $l_1^a = l_2^b = \cdots = 0$ is excluded from the summation. Substituting Eq. (17) into Eqs. (15) and (16) yields the generalized cluster expansions of forms from Eq. (4):

$$\frac{P}{k_B T} = \sum_{L=1}^{\infty} \sum_{\Sigma_{i,a} l_i^a = L} b[\{l_i^a\}] \prod_{j,b} z_b^{l_j^b}, \tag{18}$$

$$\frac{N}{V} = \sum_{L=1}^{\infty} \sum_{\Sigma_{i,a} l_i^a = L} L b[\{l_i^a\}] \prod_{j,b} z_b^{l_j^b}, \tag{19}$$

$$b[l_i^a] = \frac{1}{V} \sum_{i,a} G_i^a c_i^a[\{l_i^a\}] e^{-\beta \epsilon_i^a}. \tag{20}$$

In this way, once the $c_i^a[\{l_i^a\}]$ are obtained from the set of functional equations among the other species with the MFES [Eq. (14)], then the problem can be solved. This method can be thought of as a generalization of the method first adopted by Sutherland[8] for the CSM (Refs. 7 and 8) in one dimension where only one variable α appears for the pure FES to applied systems with MFES where many variables α_i^a appear.

For example, in the case of the system of an M-species ideal gas with MFES where the statistical parameters are defined by $g_{ij}^{ab} = g_{ab} \delta_{ij}$, one can define as $\zeta_i^a = \zeta_a(\mathbf{p})$ and the particle energy as $\epsilon_i^a = \epsilon_a(\mathbf{p}) = \mathbf{p}^2/2m_a$ for $a = 1, \ldots, M$. Then, by using a good quantum number $\mathbf{p}$, Eqs. (14)–(16) are represented as

$$[\zeta_a(\mathbf{p})^{g_{aa}} - \zeta_a(\mathbf{p})^{g_{aa}-1}] \prod_{b \neq a} \zeta_b(\mathbf{p})^{g_{ba}} = e^{\beta[\mu_a - \epsilon_a(\mathbf{p})]} \equiv \alpha_a(\mathbf{p}), \tag{21}$$

$$\frac{P}{k_B T} = \frac{1}{V} \sum_{a,\mathbf{p}} \ln \zeta_a(\mathbf{p}), \tag{22}$$

$$\frac{N}{V} = \sum_a \frac{N_a}{V} = \sum_a z_a \frac{\partial}{\partial z_a} \frac{1}{V} \sum_{\mathbf{p}} \ln \zeta_a(\mathbf{p}). \tag{23}$$

Therefore, we obtain the cluster expansions as

$$\frac{P}{k_B T} = \sum_{L=1}^{\infty} \sum_{\Sigma_a l_a = L} b[\{l_a\}] \prod_b z_b^{l_b}, \tag{24}$$

$$\frac{N}{V} = \sum_{L=1}^{\infty} \sum_{\Sigma_a l_a = L} L b[\{l_i^a\}] \prod_b z_b^{l_b}, \tag{25}$$

$$b[\{l_a\}] = \frac{1}{V} \sum_{a,\mathbf{p}} c_a[\{l_a\}] e^{-\beta \epsilon_a(\mathbf{p})}. \tag{26}$$

V. GENERALIZATION OF THE LAGRANGE THEOREM

From this section we are going to explicitly obtain the cluster coefficients $c_a[\{l_a\}]$. For this purpose, we first present the generalization of the Lagrange theorem for the Lagrange series expansion for the one complex variable functions[23] to the one for the many complex variable functions.[26]

A. Cauchy theorem for many complex variable functions

Denote by $z = (z_1, z_2, \ldots, z_M)$ the set of complex variables $z_1, z_2, \ldots, z_M$. This z defines an M-torus, $\mathbf{C}^M$. Define the M-disk of $C^M = C_1 \times C_2 \times \cdots \times C_M$ such that $|z_a - p_a| < r_a$, $a = 1, \ldots, M$,[26] where p_a denote constants. Denote by $f(z)$ an analytic function of many complex variables, which is defined over a domain $D \subset \mathbf{C}^M$. Then, one has the generalized Cauchy theorem:[26]

$$f(z) = \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{f(\zeta) d\zeta_1 d\zeta_2 \cdots d\zeta_M}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_M - z_M)}. \tag{27}$$

This provides a Taylor expansion of $f(z)$:

$$f(z) = \sum_{L=1}^{\infty} \sum_{l_1 + l_2 + \cdots + l_M = L} f_{l_1 l_2 \cdots l_M} z_1^{l_1} \cdots z_M^{l_M}, \tag{28}$$

$$f_{l_1 l_2 \cdots l_M} = \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{f(\zeta)\, d\zeta_1 d\zeta_2 \cdots d\zeta_M}{\zeta_1^{l_1+1} \zeta_2^{l_2+1} \cdots \zeta_M^{l_M+1}} = \frac{1}{l_1! l_2! \cdots l_M!} \frac{\partial^{l_1+l_2+\cdots+l_M}}{\partial z_1^{l_1} \partial z_2^{l_2} \cdots \partial z_M^{l_M}} f(0,0,\ldots,0). \tag{29}$$

B. Generalized Cauchy theorem

Let us generalize the Cauchy theorem for a set of analytic functions. Let us define a set of analytic functions of M complex variables as $F_a(z) (a=1,\ldots,M)$, which are defined over a domain $D \subset \mathbf{C}^M$. Suppose that there exist the set of zeros such that $F_a(z)=0 \quad (a=1,\ldots,M)$. Then, we have the generalized Cauchy theorem:

$$f(z) = \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{f(\zeta)}{F_1(\zeta) F_2(\zeta) \cdots F_M(\zeta)} dF_1 dF_2 \cdots dF_M \tag{30}$$

$$= \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{f(\zeta)}{F_1(\zeta) F_2(\zeta) \cdots F_M(\zeta)} |J_M| d\zeta_1 d\zeta_2 \cdots d\zeta_M, \tag{31}$$

where $|J_M|$ means the Jacobian, which is defined by

$$|J_M| = \left| \frac{\partial(F_1, F_2, \ldots, F_M)}{\partial(\zeta_1, \zeta_2, \ldots, \zeta_M)} \right| = \begin{vmatrix} \frac{\partial F_1(\zeta)}{\partial \zeta_1} & \frac{\partial F_1(\zeta)}{\partial \zeta_2} & \cdots & \frac{\partial F_1(\zeta)}{\partial \zeta_M} \\ \frac{\partial F_2(\zeta)}{\partial \zeta_1} & \frac{\partial F_2(\zeta)}{\partial \zeta_2} & \cdots & \frac{\partial F_2(\zeta)}{\partial \zeta_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_M(\zeta)}{\partial \zeta_1} & \frac{\partial F_M(\zeta)}{\partial \zeta_2} & \cdots & \frac{\partial F_M(\zeta)}{\partial \zeta_M} \end{vmatrix}. \tag{32}$$

C. Generalized Lagrange theorem

Let us next derive a generalized Lagrange theorem. Define the set of analytic functions of many complex variables as $F_a(z)$ for $a=1,\ldots,M$, which are defined over a domain $D \subset \mathbf{C}^M$. Assume that $F_a(\zeta)=x_a (a=1,\ldots,M)$ are satisfied at z and give simple roots for each variable of z_a. Suppose that there exist the roots of $F_a(z)=0$ such that the roots are single for each variable of z_a. If we assume $|F_a(\zeta)| \geq |x_a|$, then the roots of $F_a(z)=x_a$ do not change too much from the roots of $F_a(z)=0$. Therefore, this situation can be regarded as a generalized Rouché theorem.[23] If this is true, then Eq. (31) can be generalized to

$$g(z) = \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{g(\zeta)}{[F_1(\zeta)-x_1][F_2(\zeta)-x_2] \cdots [F_M(\zeta)-x_M]} dF_1 dF_2 \cdots dF_M \tag{33}$$

$$= \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{g(\zeta)}{[F_1(\zeta)-x_1][F_2(\zeta)-x_2] \cdots [F_M(\zeta)-x_M]} |J_M| d\zeta_1 d\zeta_2 \cdots d\zeta_M, \tag{34}$$

where the Jacobian $|J_M|$ is defined by

$$|J_M| = \left| \frac{\partial(F_1 - x_1, F_2 - x_2, \ldots, F_M - x_M)}{\partial(\zeta_1, \zeta_2, \ldots, \zeta_M)} \right| = \left| \frac{\partial(F_1, F_2, \ldots, F_M)}{\partial(\zeta_1, \zeta_2, \ldots, \zeta_M)} \right|. \tag{35}$$

Let us use this to generalize the Lagrange theorem for inverting the functions. Define

$$F_a(\zeta) = \frac{\zeta_a - p_a}{\phi_a(\zeta)} \quad (a=1,\ldots,M), \tag{36}$$

such that Rouché's condition is satisfied (i.e., $|x_a \phi_a| \leq |\zeta_a - p_a|, a=1,\ldots,M$), where p_a are all constants. Hence, the set of equations $F_a(\zeta)=x_a \quad (a=1,\ldots,M)$ gives

$$\zeta_a = p_a + x_a \phi_a(\zeta) \quad (a=1,\ldots,M), \tag{37}$$

which define x_a as the functions of ζ such that $x_a = x_a(\zeta) (a=1,\ldots,M)$. Let us invert Eq. (37) as $\zeta_a = \zeta_a(x)$, which is known as the Lagrange series expansion.[23] If the above Rouché's condition is satisfied, then we find

$$\frac{\partial F_a(\zeta)}{\partial \zeta_b} = \frac{\partial}{\partial \zeta_b}\left(\frac{\zeta_a - p_a}{\phi_a(\zeta)}\right) = \frac{\delta_{ab}\phi_a - (\zeta_a - p_a)\dfrac{\partial \phi_a}{\partial \zeta_b}}{\phi_a^2} = \frac{\delta_{ab} - x_a \dfrac{\partial \phi_a}{\partial \zeta_b}}{\phi_a}, \tag{38}$$

where in the last step we have used the relation $\zeta_a = p_a + x_a\phi_a(\zeta)$. Substituting Eq. (38) into Eq. (35), the Jacobian becomes

$$|J_M| = \begin{vmatrix} \dfrac{1 - x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_1}}{\phi_1} & \dfrac{-x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_2}}{\phi_1} & \cdots & \dfrac{-x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_M}}{\phi_1} \\ \dfrac{-x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_1}}{\phi_2} & \dfrac{1 - x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_2}}{\phi_2} & \cdots & \dfrac{-x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_M}}{\phi_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{-x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_1}}{\phi_M} & \dfrac{-x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_2}}{\phi_M} & \cdots & \dfrac{1 - x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_M}}{\phi_M} \end{vmatrix}$$

$$= \frac{1}{\phi_1\phi_2\cdots\phi_M} \begin{vmatrix} 1 - x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_1} & -x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_2} & \cdots & -x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_M} \\ -x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_1} & 1 - x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_2} & \cdots & -x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_M} \\ \vdots & \vdots & \ddots & \vdots \\ -x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_1} & -x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_2} & \cdots & 1 - x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_M} \end{vmatrix}. \tag{39}$$

On the other hand, the denominator of Eq. (34) becomes

$$[F_1(\zeta) - x_1][F_2(\zeta) - x_2]\cdots[F_M(\zeta) - x_M] = \frac{1}{\phi_1\phi_2\cdots\phi_M}[\zeta_1 - p_1 - x_1\phi_1(\zeta)][\zeta_2 - p_2 - x_2\phi_2(\zeta)]\cdots[\zeta_M - p_M - x_M\phi_M(\zeta)]. \tag{40}$$

Hence, we have

$$g(z) = \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{g(\zeta)|J_M|}{(\zeta_1 - p_1 - x_1\phi_1)(\zeta_2 - p_2 - x_2\phi_2)\cdots(\zeta_M - p_M - x_M\phi_M)} d\zeta_1 d\zeta_2 \cdots d\zeta_M, \tag{41}$$

$$|J_M| = \begin{vmatrix} 1 - x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_1} & -x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_2} & \cdots & -x_1\dfrac{\partial\phi_1(\zeta)}{\partial\zeta_M} \\ -x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_1} & 1 - x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_2} & \cdots & -x_2\dfrac{\partial\phi_2(\zeta)}{\partial\zeta_M} \\ \vdots & \vdots & \ddots & \vdots \\ -x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_1} & -x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_2} & \cdots & 1 - x_M\dfrac{\partial\phi_M(\zeta)}{\partial\zeta_M} \end{vmatrix}. \tag{42}$$

Let us expand both the enumerator and denominator of Eq. (41) with respect to x_a $(a = 1, \dots, M)$. Here, we can expand the denominator as

$$\frac{1}{\zeta_a - p_a - x_a\phi_a} = \sum_{l_a=0}^{\infty} \frac{\phi_a^{l_a} x_a^{l_a}}{(\zeta_a - p_a)^{l_a+1}}. \tag{43}$$

Hence, we have

$$\frac{1}{(\zeta_1 - p_1 - x_1\phi_1)(\zeta_2 - p_2 - x_2\phi_2)\cdots(\zeta_M - p_M - x_M\phi_M)} = \sum_{\{l_a\}=0}^{\infty} \frac{\phi_1^{l_1}\phi_2^{l_2}\cdots\phi_M^{l_M}}{(\zeta_1 - p_1)^{l_1+1}(\zeta_2 - p_2)^{l_2+1}\cdots(\zeta_M - p_M)^{l_M+1}} x_1^{l_1} x_2^{l_2}\cdots x_M^{l_M}. \tag{44}$$

The determinant in the enumerator is expanded as

$$|J_M|=1-\sum_{j=1}^{M} x_j\frac{\partial\phi_j}{\partial\zeta_j}+\sum_{j<k=1}^{M} x_jx_k\left|\frac{\partial(\phi_j,\phi_k)}{\partial(\zeta_j,\zeta_k)}\right|+\cdots+(-1)^M\ x_1x_2\cdots x_M\left|\frac{\partial(\phi_1,\phi_2,\ldots,\phi_M)}{\partial(\zeta_1,\zeta_2,\ldots,\zeta_M)}\right|. \tag{45}$$

Substituting Eqs. (44) and (45) into Eq. (41), we find

$$g(z)=\sum_{L=1}^{\infty}\sum_{l_1+l_2+\cdots+l_M=L} g_{l_1l_2\cdots l_M}x_1^{l_1}x_2^{l_2}\cdots x_M^{l_M}, \tag{46}$$

where

$$g_{l_1\cdots l_M}=K^{(0)}_{l_1\cdots l_M}-\sum_{j=1}^{M} K^{(1)}_{l_1\cdots l_j-1\cdots l_M}+\sum_{j<k=1}^{M} K^{(2)}_{l_1\cdots l_j-1\cdots l_k-1\cdots l_M}+\cdots+(-1)^MK^{(M)}_{l_1-1\cdots l_M-1}. \tag{47}$$

Here we have defined as

$$K^{(0)}_{l_1\cdots l_M}=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}g(\zeta)\frac{\phi_1(\zeta)^{l_1}\cdots\phi_M(\zeta)^{l_M}}{(\zeta_1-p_1)^{l_1+1}\cdots(\zeta_M-p_M)^{l_M+1}}d\zeta_1\cdots d\zeta_M, \tag{48a}$$

$$K^{(1)}_{l_1\cdots l_j-1\cdots l_M}=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}\frac{g(\zeta)\phi_1(\zeta)^{l_1}\cdots\phi_j(\zeta)^{l_j-1}\cdots\phi_M(\zeta)^{l_M}\dfrac{\partial\phi_j}{\partial\zeta_j}}{(\zeta_1-p_1)^{l_1+1}\cdots(\zeta_j-p_j)^{l_j}\cdots(\zeta_M-p_M)^{l_M+1}}d\zeta_1\cdots d\zeta_M, \tag{48b}$$

$$K^{(2)}_{l_1\cdots l_j-1\cdots l_k-1\cdots l_M}=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}\frac{g(\zeta)\phi_1(\zeta)^{l_1}\cdots\phi_j(\zeta)^{l_j-1}\cdots\phi_k(\zeta)^{l_k-1}\cdots\phi_M(\zeta)^{l_M}}{(\zeta_1-p_1)^{l_1+1}\cdots(\zeta_j-p_j)^{l_j}\cdots(\zeta_k-p_k)^{l_k}\cdots(\zeta_M-p_M)^{l_M+1}}\left|\frac{\partial(\phi_j,\phi_k)}{\partial(\zeta_j,\zeta_k)}\right|d\zeta_1\cdots d\zeta_M, \tag{48c}$$

$$\vdots$$

$$K^{(M)}_{l_1-1\cdots l_M-1}=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}\frac{g(\zeta)\phi_1(\zeta)^{l_1-1}\cdots\phi_M(\zeta)^{l_M-1}}{(\zeta_1-p_1)^{l_1}\cdots(\zeta_M-p_M)^{l_M}}\left|\frac{\partial(\phi_1,\ldots,\phi_M)}{\partial(\zeta_1,\ldots,\zeta_M)}\right|d\zeta_1\cdots d\zeta_M, \tag{48m}$$

respectively. From the generalized Cauchy theorem we find the following relations:

$$g(p)\phi_1(p)^{l_1}\cdots\phi_M(p)^{l_M}=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}g(\zeta)\frac{\phi_1(\zeta)^{l_1}\cdots\phi_M(\zeta)^{l_M}}{(\zeta_1-p_1)\cdots(\zeta_M-p_M)}d\zeta_1\cdots d\zeta_M, \tag{49a}$$

$$g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_M(p)^{l_M}\frac{\partial\phi_j}{\partial p_j}=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}g(\zeta)\frac{\phi_1(\zeta)^{l_1}\cdots\phi_j(\zeta)^{l_j-1}\cdots\phi_M(\zeta)^{l_M}\dfrac{\partial\phi_j}{\partial\zeta_j}}{(\zeta_1-p_1)\cdots(\zeta_j-p_j)\cdots(\zeta_M-p_M)}d\zeta_1\cdots d\zeta_M, \tag{49b}$$

$$g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_k(p)^{l_k-1}\cdots\phi_M(p)^{l_M}\left|\frac{\partial(\phi_j,\phi_k)}{\partial(p_j,p_k)}\right|$$
$$=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}g(\zeta)\frac{\phi_1(\zeta)^{l_1}\cdots\phi_j(\zeta)^{l_j-1}\cdots\phi_k(\zeta)^{l_k-1}\cdots\phi_M(\zeta)^{l_M}\left|\dfrac{\partial(\phi_j,\phi_k)}{\partial(\zeta_j,\zeta_k)}\right|}{(\zeta_1-p_1)\cdots(\zeta_j-p_j)\cdots(\zeta_k-p_k)\cdots(\zeta_M-p_M)}d\zeta_1\cdots d\zeta_M, \tag{49c}$$

$$\vdots$$

$$g(p)\phi_1(p)^{l_1-1}\cdots\phi_M(p)^{l_M-1}\left|\frac{\partial(\phi_1,\cdots,\phi_M)}{\partial(p_1,\cdots,p_M)}\right|$$
$$=\left(\frac{1}{2\pi i}\right)^M\int_{C^M}g(\zeta)\frac{\phi_1(\zeta)^{l_1-1}\cdots\phi_M(\zeta)^{l_M-1}\left|\dfrac{\partial(\phi_1,\cdots,\phi_M)}{\partial(\zeta_1,\cdots,\zeta_M)}\right|}{(\zeta_1-p_1)\cdots(\zeta_M-p_M)}d\zeta_1\cdots d\zeta_M, \tag{49m}$$

Using these together with Eq. (29), we obtain

$$K^{(0)}_{l_1\cdots l_M}=\frac{1}{l_1!\cdots l_M!}\,\frac{\partial^{l_1+\cdots+l_M}}{\partial p_1^{l_1}\cdots\partial p_M^{l_M}}[g(p)\phi_1(p)^{l_1}\cdots\phi_M(p)^{l_M}], \tag{50a}$$

$$K^{(1)}_{l_1\cdots l_j-1\cdots l_M}=\frac{l_j}{l_1!\cdots l_M!}\,\frac{\partial^{l_1+\cdots+l_j\cdots+l_M-1}}{\partial p_1^{l_1}\cdots\partial p_j^{l_j-1}\cdots\partial p_M^{l_M}}\left(g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_M(p)^{l_M}\frac{\partial\phi_j}{\partial p_j}\right), \tag{50b}$$

$$\begin{aligned}K^{(2)}_{l_1\cdots l_j\cdots l_k\cdots l_M}&\\ =\frac{l_jl_k}{l_1!\cdots l_M!}\,\frac{\partial^{l_1+\cdots+l_j+\cdots+l_k+\cdots+l_M-2}}{\partial p_1^{l_1}\cdots\partial p_j^{l_j-1}\cdots\partial p_k^{l_k-1}\cdots\partial p_M^{l_M}}&\left(g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_k(p)^{l_k-1}\cdots\phi_M(p)^{l_M}\left|\frac{\partial(\phi_j,\phi_k)}{\partial(p_j,p_k)}\right|\right),\end{aligned} \tag{50c}$$

$$\vdots$$

$$K^{(M)}_{l_1\cdots l_M}=\frac{l_1\cdots l_M}{l_1!\cdots l_M!}\,\frac{\partial^{l_1+\cdots+l_M-M}}{\partial p_1^{l_1-1}\cdots\partial p_M^{l_M-1}}\left(g(p)\phi_1(p)^{l_1-1}\cdots\phi_M(p)^{l_M-1}\left|\frac{\partial(\phi_1,\ldots,\phi_M)}{\partial(p_1,\ldots,p_M)}\right|\right). \tag{50m}$$

The above result can be written in the following way: Let us denote $G_M(p)=G_M(p_1,p_2,\ldots,p_M)$ as

$$\begin{aligned}G_M(p)=&\,\partial_1\cdots\partial_M[g(p)\phi_1(p)^{l_1}\phi_2(p)^{l_2}\cdots\phi_M(p)^{l_M}]-\sum_{j=1}^{M}\partial_1\cdots\partial_j^*\cdots\partial_M\left(g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_M(p)^{l_M}\frac{\partial\phi_j}{\partial p_j}\right)\\&+\sum_{j<k=1}^{M}\partial_1\cdots\partial_j^*\cdots\partial_k^*\cdots\partial_M\left(g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_k(p)^{l_k-1}\cdots\phi_M(p)^{l_M}\left|\frac{\partial(\phi_j,\phi_k)}{\partial(p_j,p_k)}\right|\right)\\&\vdots\\&+(-1)^M\left(g(p)\phi_1(p)^{l_1-1}\cdots\phi_M(p)^{l_M-1}\left|\frac{\partial(\phi_1,\ldots,\phi_M)}{\partial(p_1,\ldots,p_M)}\right|\right),\end{aligned} \tag{51}$$

where $\partial_j=\partial/\partial p_j$ and ∂_j^* means elimination of ∂_j. Then we find

$$g_{l_10\cdots0}=\frac{1}{l_1!}\,\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}G_1(p_1), \tag{52a}$$

$$g_{l_1l_20\cdots0}=\frac{1}{l_1!l_2!}\,\frac{\partial^{l_1+l_2-2}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}}G_2(p_1,p_2), \tag{52b}$$

$$\vdots$$

$$g_{l_1l_2\cdots l_M}=\frac{1}{l_1!l_2!\cdots l_M!}\,\frac{\partial^{l_1+l_2+\cdots+l_M-M}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}\cdots\partial p_M^{l_M-1}}G_M(p). \tag{52m}$$

Thus, we are able to complete the inversion of the set of the functions. Hence, this is the generalization of the Lagrange theorem for the single complex variable functions to that for the multicomplex variable functions.

VI. LAGRANGE EXPANSIONS FOR SYSTEMS WITH MFES

The above formulas are still too general, so that the inversion depends on the characteristics of the functions $\phi_a(\zeta)$ $(a=1,2,\ldots,M)$. However, if the functions have a particular condition, then the problem becomes much simpler. Let us find this condition in our problem of the MFES. From Eq. (14), we divide this by $\zeta_a(\mathbf{p})^{g_{aa}-1}$. Then we get

$$\zeta_a(\mathbf{p})=1-\alpha_a(\mathbf{p})\zeta_a(\mathbf{p})^{1-g_{aa}}\prod_{b\neq a}\zeta_b(\mathbf{p})^{-g_{ba}}. \tag{53}$$

Therefore, in the case of MFES we identify as

$$p_a=1, \tag{54}$$

$$x_a=\alpha_a, \tag{55}$$

$$\phi_a(\zeta)=\zeta_a^{1-g_{aa}}\prod_{b\neq a}\zeta_b^{-g_{ba}}, \tag{56}$$

which yield

$$\frac{\partial\phi_a}{\partial\zeta_b}=\frac{\delta_{ab}-g_{ba}}{\zeta_b}\phi_a(\zeta). \tag{57}$$

From this, we find

$$\frac{\partial\phi_j}{\partial\zeta_j}=\frac{1-g_{jj}}{\zeta_j}\phi_j(\zeta). \tag{58a}$$

$$\left|\frac{\partial(\phi_j,\phi_k)}{\partial(\zeta_j,\zeta_k)}\right|=\begin{vmatrix}\dfrac{\partial\phi_j(\zeta)}{\partial\zeta_j} & \dfrac{\partial\phi_j(\zeta)}{\partial\zeta_k}\\ \dfrac{\partial\phi_k(\zeta)}{\partial\zeta_j} & \dfrac{\partial\phi_k(\zeta)}{\partial\zeta_k}\end{vmatrix}=\begin{vmatrix}1-g_{jj} & -g_{kj}\\ -g_{jk} & 1-g_{kk}\end{vmatrix}\frac{\phi_j(\zeta)\phi_k(\zeta)}{\zeta_j\zeta_k}, \tag{58b}$$

$$\vdots$$

$$\left|\frac{\partial(\phi_1,\cdots,\phi_M)}{\partial(\zeta_1,\cdots,\zeta_M)}\right|=\begin{vmatrix}1-g_{11} & -g_{21} & \cdots & -g_{M1}\\ -g_{12} & 1-g_{22} & \cdots & -g_{M2}\\ \vdots & \vdots & \ddots & \vdots\\ -g_{1M} & -g_{2M} & \cdots & 1-g_{MM}\end{vmatrix}\frac{\phi_1(\zeta)\cdots\phi_M(\zeta)}{\zeta_1\cdots\zeta_M}. \tag{58m}$$

Using these for Eq. (49), we find

$$g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_M(p)^{l_M}\frac{\partial\phi_j}{\partial p_j}=\frac{1-g_{jj}}{p_j}g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j}\cdots\phi_M(p)^{l_M}, \tag{59a}$$

$$\begin{aligned}&g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j-1}\cdots\phi_k(p)^{l_k-1}\cdots\phi_M(p)^{l_M}\left|\frac{\partial(\phi_j,\phi_k)}{\partial(p_j,p_k)}\right|\\ &=\frac{1}{p_jp_k}\begin{vmatrix}1-g_{jj} & -g_{kj}\\ -g_{jk} & 1-g_{kk}\end{vmatrix}g(p)\phi_1(p)^{l_1}\cdots\phi_j(p)^{l_j}\cdots\phi_M(p)^{l_M},\end{aligned} \tag{59b}$$

$$\vdots$$

$$g(p)\phi_1(p)^{l_1-1}\cdots\phi_M(p)^{l_M-1}\left|\frac{\partial(\phi_1,\cdots,\phi_M)}{\partial(p_1,\cdots,p_M)}\right|=\frac{1}{p_1\cdots p_M}\begin{vmatrix}1-g_{11} & -g_{21} & \cdots & -g_{M1}\\ -g_{12} & 1-g_{22} & \cdots & -g_{M2}\\ \vdots & \vdots & \ddots & \vdots\\ -g_{1M} & -g_{2M} & \cdots & 1-g_{MM}\end{vmatrix}g(p)\phi_1(p)^{l_1}\cdots\phi_M(p)^{l_M}. \tag{59m}$$

On the other hand, we also have

$$\begin{aligned}\phi_1(p)^{l_1}\phi_2(p)^{l_2}\cdots\phi_M(p)^{l_M}&=(p_1^{1-g_{11}}p_2^{-g_{21}}\cdots p_M^{-g_{M1}})^{l_1}(p_1^{-g_{12}}p_2^{1-g_{22}}\cdots p_M^{-g_{M2}})^{l_2}\cdots(p_1^{-g_{1M}}p_2^{-g_{2M}}\cdots p_M^{1-g_{MM}})^{l_M}\\ &=p_1^{s_1}p_2^{s_2}\cdots p_M^{s_M},\end{aligned} \tag{60}$$

where we have defined as

$$s_1=l_1(1-g_{11})-l_2g_{12}-\cdots-l_Mg_{1M}, \tag{61a}$$

$$s_2=-l_1g_{21}+l_2(1-g_{22})-\cdots-l_Mg_{2M}, \tag{61b}$$

$$\vdots$$

$$s_M=-l_1g_{M1}-l_2g_{M2}-\cdots+l_M(1-g_{MM}), \tag{61m}$$

respectively. Substituting Eq. (59) into Eq. (51), we find

$$\begin{aligned}G_M(p)=&\partial_1\cdots\partial_M[g(p)p_1^{s_1}p_2^{s_2}\cdots p_M^{s_M}]-\sum_{j=1}^{M} l_j(1-g_{jj})\partial_1\cdots\partial_j^*\cdots\partial_M[g(p)p_1^{s_1}\cdots p_j^{s_j-1}\cdots p_M^{s_M}]\\&+\sum_{j<k=1}^{M}\begin{vmatrix} l_j(1-g_{jj}) & -l_jg_{kj}\\ -l_kg_{jk} & l_k(1-g_{kk})\end{vmatrix}\partial_1\cdots\partial_j^*\cdots\partial_k^*\cdots\partial_M[g(p)p_1^{s_1}\cdots p_j^{s_j-1}\cdots p_k^{s_k-1}\cdots p_M^{s_M}]\\&\vdots\\&+(-1)^M\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} & \cdots & -l_1g_{M1}\\ -l_2g_{12} & l_2(1-g_{22}) & \cdots & -l_2g_{M2}\\ \vdots & \vdots & \ddots & \vdots\\ -l_Mg_{1M} & -l_Mg_{2M} & \cdots & l_M(1-g_{MM})\end{vmatrix}[g(p)p_1^{s_1-1}\cdots p_M^{s_M-1}].\end{aligned}\tag{62}$$

In our physical systems of MFES, we are concerned with the thermodynamic potential of Eq. (22). Therefore, the $g(p)$ is given by

$$g(p)=\sum_{a=1}^{M}\ln p_a.\tag{63}$$

From this, if we regard p_a as parameters, then we find the following properties of $g(p)$:

$$\partial_j g(p)=\frac{1}{p_j},\quad \partial_j\partial_k g(p)=0,\partial_i\partial_j\partial_k g(p)=0,\quad \text{etc.}\tag{64}$$

Using these properties, we can calculate Eq. (62) as follows. Let us present here some examples to see how the above formulas work.

Let us first consider the case of $M=1$. In this case, we have the following:

$$g_{l_1}=\frac{1}{l_1!}\left(\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}G_1(p_1)\right)_{p_1=1},\tag{65}$$

where $G_1(p_1)$ is defined by

$$G_1(p_1)=\partial_1[g(p_1)p_1^{s_1}]-g(p_1)\frac{l_1(1-g_{11})}{p_1}p_1^{s_1},\tag{66}$$

with $s_1=l_1(1-g_{11})$. This can be simplified as

$$G_1(p_1)=\partial_1[g(p_1)]p_1^{s_1}+g(p_1)\partial_1 p_1^{s_1}-g(p_1)l_1(1-g_{11})p_1^{s_1-1}=\partial_1[g(p_1)]p_1^{s_1},\tag{67}$$

where in the last step we have used the relation $s_1=l_1(1-g_{11})$. Since $g(p_1)=\ln p_1$, $\partial_1 g(p_1)=1/p_1$, by substituting Eq. (64) into Eq. (67) we get $G_1(p_1)=p_1^{s_1-1}$. Hence, substituting this into Eq. (65) yields

$$\begin{aligned}g_{l_1}&=\frac{1}{l_1!}\left(\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}=\frac{1}{l_1!}\left(\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{l_1(1-g_{11})-1}\right)_{p_1=1}\\&=\frac{1}{l_1!}[l_1(1-g_{11})-1][l_1(1-g_{11})-2]\cdots[l_1(1-g_{11})-(l_1-1)]\\&=\frac{1}{l_1!}(l_1-1-l_1g_{11})(l_1-2-l_1g_{11})\cdots(1-l_1g_{11}).\end{aligned}\tag{68}$$

Second, let us consider the case of $M=2$. In this case, we have the following:

$$g_{l_1l_2}=\frac{1}{l_1!l_2!}\left(\frac{\partial^{l_1+l_2-2}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}}G_2(p)\right)_{p_1=p_2=1}.\tag{69}$$

Here the $G_2(p)$ is defined by

$$G_2(p)=\partial_1\partial_2[g(p)p_1^{s_1}p_2^{s_2}]-l_1(1-g_{11})\partial_2[g(p)p_1^{s_1-1}p_2^{s_2}]-l_2(1-g_{22})\partial_1[g(p)p_1^{s_1}p_2^{s_2-1}]$$
$$+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} \\ -l_2g_{12} & l_2(1-g_{22}) \end{vmatrix} g(p)p_1^{s_1-1}p_2^{s_2-1}, \tag{70}$$

where we have defined as

$$s_1=l_1(1-g_{11})-l_2g_{12}, \tag{71a}$$

$$s_2=-l_1g_{21}+l_2(1-g_{22}), \tag{71b}$$

$$g(p)=\ln p_1+\ln p_2. \tag{72}$$

By using the relations of Eq. (64), we obtain

$$G_2(p)=p_1^{s_1-1}p_2^{s_2-1}[A_2+g(p)B_2], \tag{73}$$

$$A_2=s_1+s_2-l_1(1-g_{11})-l_2(1-g_{22}), \tag{74}$$

$$B_2=s_1s_2-l_1(1-g_{11})s_2-l_2(1-g_{22})s_1+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} \\ -l_2g_{12} & l_2(1-g_{22}) \end{vmatrix}. \tag{75}$$

By direct calculations of A_2 and B_2, we obtain

$$A_2=-(l_1g_{21}+l_2g_{12}), \tag{76}$$

$$B_2=\begin{vmatrix} l_1(1-g_{11})-s_1 & -l_1g_{21} \\ -l_2g_{12} & l_2(1-g_{22})-s_2 \end{vmatrix}=\begin{vmatrix} l_1g_{21} & -l_1g_{21} \\ -l_2g_{12} & l_2g_{12} \end{vmatrix}=0. \tag{77}$$

Hence, substituting Eq. (73) into Eq. (69) yields

$$\begin{aligned} g_{l_1l_2}&=-(l_1g_{21}+l_2g_{12})\frac{1}{l_1!l_2!}\left(\frac{\partial^{l_1+l_2-2}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}}p_1^{s_1-1}p_2^{s_2-1}\right)_{p_1=p_2=1} \\ &=-(l_1g_{21}+l_2g_{12})\left(\frac{1}{l_1!}\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}\left(\frac{1}{l_2!}\frac{\partial^{l_2-1}}{\partial p_2^{l_2-1}}p_2^{s_2-1}\right)_{p_2=1} \\ &=-(l_1g_{21}+l_2g_{12})\frac{1}{l_1!l_2!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)](s_2-1)(s_2-2)\cdots[s_2-(l_2-1)]. \end{aligned} \tag{78}$$

On the other hand, by following a similar procedure we find

$$g_{l_10}=\frac{1}{l_1!}\left(\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}=\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)], \tag{79}$$

$$g_{0l_2}=\frac{1}{l_2!}\left(\frac{\partial^{l_2-1}}{\partial p_2^{l_2-1}}p_2^{s_2-1}\right)_{p_2=1}=\frac{1}{l_2!}(s_2-1)(s_2-2)\cdots[s_2-(l_2-1)]. \tag{80}$$

Third, let us consider the case of $M=3$. In this case, we have the following:

$$g_{l_1l_2l_3}=\frac{1}{l_1!l_2!l_3!}\left(\frac{\partial^{l_1+l_2+l_3-3}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}\partial p_3^{l_3-1}}G_3(p)\right)_{p_1=p_2=p_3=1}. \tag{81}$$

Here the $G_3(p)$ is defined by

$$
\begin{aligned}
G_3(p)={}&\partial_1\partial_2\partial_3[g(p)p_1^{s_1}p_2^{s_2}p_3^{s_3}]-l_1(1-g_{11})\partial_2\partial_3[g(p)p_1^{s_1-1}p_2^{s_2}p_3^{s_3}]-l_2(1-g_{22})\partial_1\partial_3[g(p)p_1^{s_1}p_2^{s_2-1}p_3^{s_3}]\\
&-l_3(1-g_{33})\partial_1\partial_2[g(p)p_1^{s_1}p_2^{s_2}p_3^{s_3-1}]+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21}\\ -l_2g_{12} & l_2(1-g_{22})\end{vmatrix}\partial_3[g(p)p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{31}\\ -l_3g_{13} & l_3(1-g_{33})\end{vmatrix}\partial_2[g(p)p_1^{s_1-1}p_2^{s_2}p_3^{s_3-1}]+\begin{vmatrix} l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}\partial_1[g(p)p_1^{s_1}p_2^{s_2-1}p_3^{s_3-1}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{31}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}g(p)p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1}, \qquad (82)
\end{aligned}
$$

where we have defined as

$$s_1=l_1(1-g_{11})-l_2g_{12}-l_3g_{13}, \qquad (83a)$$

$$s_2=-l_1g_{21}+l_2(1-g_{22})-l_3g_{23}, \qquad (83b)$$

$$s_3=-l_1g_{31}-l_2g_{32}+l_3(1-g_{33}), \qquad (83c)$$

$$g(p)=\ln p_1+\ln p_2+\ln p_3. \qquad (84)$$

After a rather tedious algebra, the above equation is given by

$$G_3(p)=p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1}[A_3+g(p)B_3], \qquad (85)$$

where A_3 and B_3 are given by

$$
\begin{aligned}
A_3={}&s_1s_2+s_1s_3+s_2s_3-l_1(1-g_{11})(s_2+s_3)-l_2(1-g_{22})(s_1+s_3)-l_3(1-g_{33})(s_1+s_2)+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21}\\ -l_2g_{12} & l_2(1-g_{22})\end{vmatrix}\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{31}\\ -l_3g_{13} & l_3(1-g_{33})\end{vmatrix}+\begin{vmatrix} l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}, \qquad (86)
\end{aligned}
$$

$$
\begin{aligned}
B_3={}&s_1s_2s_3-l_1(1-g_{11})s_2s_3-l_2(1-g_{22})s_1s_3-l_3(1-g_{33})s_1s_2+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21}\\ -l_2g_{12} & l_2(1-g_{22})\end{vmatrix}s_3+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{31}\\ -l_3g_{13} & l_3(1-g_{33})\end{vmatrix}s_2\\
&+\begin{vmatrix} l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}s_1-\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{31}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}. \qquad (87)
\end{aligned}
$$

At a first glance, A_3 and B_3 look very complicated. However, comparing the above expressions to those of A_2 and B_2, Eqs. (86) and (87) are simplified as follows:

$$A_3=\begin{vmatrix} l_1(1-g_{11})-s_1 & -l_1g_{21}\\ -l_2g_{12} & l_2(1-g_{22})-s_2\end{vmatrix}+\begin{vmatrix} l_1(1-g_{11})-s_1 & -l_1g_{31}\\ -l_3g_{13} & l_3(1-g_{33})-s_3\end{vmatrix}+\begin{vmatrix} l_2(1-g_{22})-s_2 & -l_2g_{32}\\ -l_3g_{23} & l_3(1-g_{33})-s_3\end{vmatrix}, \qquad (88)$$

$$B_3=\begin{vmatrix} l_1(1-g_{11})-s_1 & -l_1g_{21} & -l_1g_{31}\\ -l_2g_{12} & l_2(1-g_{22})-s_2 & -l_2g_{32}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})-s_3\end{vmatrix}. \qquad (89)$$

Substituting the relations for s_1, s_2, and s_3 [Eq. (83)] into Eqs. (88) and (89) and expanding them with respect to l_1, l_2, and l_3, we obtain

$$
\begin{aligned}
A_3={}&\begin{vmatrix} l_2g_{12}+l_3g_{13} & -l_1g_{21}\\ -l_2g_{12} & l_1g_{21}+l_3g_{23}\end{vmatrix}+\begin{vmatrix} l_2g_{12}+l_3g_{13} & -l_1g_{31}\\ -l_3g_{13} & l_1g_{31}+l_2g_{32}\end{vmatrix}+\begin{vmatrix} l_1g_{21}+l_3g_{23} & -l_2g_{32}\\ -l_3g_{23} & l_1g_{31}+l_2g_{32}\end{vmatrix}\\
={}&l_1^2g_{21}g_{31}+l_1l_2g_{21}g_{32}+l_1l_3g_{23}g_{31}+l_1l_2g_{31}g_{12}+l_2^2g_{32}g_{12}+l_2l_3g_{32}g_{13}+l_1l_3g_{13}g_{21}+l_2l_3g_{12}g_{23}+l_3^2g_{13}g_{23}, \qquad (90)
\end{aligned}
$$

$$B_3=\begin{vmatrix} l_2g_{12}+l_3g_{13} & -l_1g_{21} & -l_1g_{31} \\ -l_2g_{12} & l_1g_{21}+l_3g_{23} & -l_2g_{32} \\ -l_3g_{13} & -l_3g_{23} & l_1g_{31}+l_2g_{32} \end{vmatrix}=0. \tag{91}$$

Substituting Eq. (85) together with Eqs. (90) and (91) into Eq. (81), we get

$$\begin{aligned} g_{l_1l_2l_3}&=A_3\frac{1}{l_1!l_2!l_3!}\left(\frac{\partial^{l_1+l_2+l_3-3}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}\partial p_3^{l_3-1}}p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1}\right)_{p_1=p_2=p_3=1} \\ &=A_3\left(\frac{1}{l_1!}\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}\left(\frac{1}{l_2!}\frac{\partial^{l_2-1}}{\partial p_2^{l_2-1}}p_2^{s_2-1}\right)_{p_2=1}\left(\frac{1}{l_3!}\frac{\partial^{l_3-1}}{\partial p_3^{l_3-1}}p_3^{s_3-1}\right)_{p_3=1} \\ &=A_3\left(\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)]\right)\left(\frac{1}{l_2!}(s_2-1)(s_2-2)\cdots[s_2-(l_2-1)]\right) \\ &\quad\times\left(\frac{1}{l_3!}(s_3-1)(s_3-2)\cdots[s_3-(l_3-1)]\right). \end{aligned} \tag{92}$$

On the other hand, we also find similarly the following:

$$g_{l_100}=\frac{1}{l_1!}\left(\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}=\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)], \tag{93a}$$

$$g_{0l_20}=\frac{1}{l_2!}\left(\frac{\partial^{l_2-1}}{\partial p_2^{l_2-1}}p_2^{s_2-1}\right)_{p_2=1}=\frac{1}{l_2!}(s_2-1)(s_2-2)\cdots[s_2-(l_2-1)], \tag{93b}$$

$$g_{00l_3}=\frac{1}{l_3!}\left(\frac{\partial^{l_3-1}}{\partial p_3^{l_3-1}}p_3^{s_3-1}\right)_{p_3=1}=\frac{1}{l_3!}(s_3-1)(s_3-2)\cdots[s_3-(l_3-1)], \tag{93c}$$

$$\begin{aligned} g_{l_1l_20}&=-(l_1g_{21}+l_2g_{12})\left(\frac{1}{l_1!}\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}\left(\frac{1}{l_2!}\frac{\partial^{l_2-1}}{\partial p_2^{l_2-1}}p_2^{s_2-1}\right)_{p_2=1} \\ &=-(l_1g_{21}+l_2g_{12})\left(\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)]\right)\left(\frac{1}{l_2!}(s_2-1)(s_2-2)\cdots[s_2-(l_2-1)]\right), \end{aligned} \tag{94a}$$

$$\begin{aligned} g_{l_10l_3}&=-(l_1g_{31}+l_3g_{13})\left(\frac{1}{l_1!}\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}\left(\frac{1}{l_3!}\frac{\partial^{l_3-1}}{\partial p_3^{l_3-1}}p_3^{s_3-1}\right)_{p_3=1} \\ &=-(l_1g_{31}+l_3g_{13})\left(\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)]\right)\left(\frac{1}{l_3!}(s_3-1)(s_3-2)\cdots[s_3-(l_3-1)]\right), \end{aligned} \tag{94b}$$

$$\begin{aligned} g_{0l_2l_3}&=-(l_2g_{32}+l_3g_{23})\left(\frac{1}{l_2!}\frac{\partial^{l_2-1}}{\partial p_2^{l_2-1}}p_2^{s_2-1}\right)_{p_2=1}\left(\frac{1}{l_3!}\frac{\partial^{l_3-1}}{\partial p_3^{l_3-1}}p_3^{s_3-1}\right)_{p_3=1} \\ &=-(l_2g_{32}+l_3g_{23})\left(\frac{1}{l_2!}(s_2-1)(s_2-2)\cdots[s_2-(l_2-1)]\right)\left(\frac{1}{l_3!}(s_3-1)(s_3-2)\cdots[s_3-(l_3-1)]\right). \end{aligned} \tag{94c}$$

Fourth, let us consider the case of $M=4$. In this case, we have the following:

$$g_{l_1l_2l_3l_4}=\frac{1}{l_1!l_2!l_3!l_4!}\left(\frac{\partial^{l_1+l_2+l_3+l_4-4}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}\partial p_3^{l_3-1}\partial p_4^{l_4-1}}G_4(p)\right)_{p_1=p_2=p_3=p_4=1}. \tag{95}$$

Here, the $G_4(p)$ is defined by

$$\begin{aligned}
G_4(p)={}&\partial_1\partial_2\partial_3\partial_4[g(p)p_1^{s_1}p_2^{s_2}p_3^{s_3}p_4^{s_4}]-l_1(1-g_{11})\partial_2\partial_3\partial_4[g(p)p_1^{s_1-1}p_2^{s_2}p_3^{s_3}p_4^{s_4}]-l_2(1-g_{22})\partial_1\partial_3\partial_4[g(p)p_1^{s_1}p_2^{s_2-1}p_3^{s_3}p_4^{s_4}]\\
&-l_3(1-g_{33})\partial_1\partial_2\partial_4[g(p)p_1^{s_1}p_2^{s_2}p_3^{s_3-1}p_4^{s_4}]-l_4(1-g_{44})\partial_1\partial_2\partial_3[g(p)p_1^{s_1}p_2^{s_2}p_3^{s_3}p_4^{s_4-1}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21}\\ -l_2g_{12} & l_2(1-g_{22})\end{vmatrix}\partial_3\partial_4[g(p)p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3}p_4^{s_4}]+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{31}\\ -l_3g_{13} & l_3(1-g_{33})\end{vmatrix}\partial_2\partial_4[g(p)p_1^{s_1-1}p_2^{s_2}p_3^{s_3-1}p_4^{s_4}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{41}\\ -l_4g_{14} & l_4(1-g_{44})\end{vmatrix}\partial_2\partial_3[g(p)p_1^{s_1-1}p_2^{s_2}p_3^{s_3}p_4^{s_4-1}]+\begin{vmatrix} l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}\partial_1\partial_4[g(p)p_1^{s_1}p_2^{s_2-1}p_3^{s_3-1}p_4^{s_4}]\\
&+\begin{vmatrix} l_2(1-g_{22}) & -l_2g_{42}\\ -l_4g_{24} & l_4(1-g_{44})\end{vmatrix}\partial_1\partial_3[g(p)p_1^{s_1}p_2^{s_2-1}p_3^{s_3}p_4^{s_4-1}]+\begin{vmatrix} l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}\partial_1\partial_2[g(p)p_1^{s_1}p_2^{s_2}p_3^{s_3-1}p_4^{s_4-1}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{31}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}\partial_4[g(p)p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1}p_4^{s_4}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{41}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{42}\\ -l_4g_{14} & -l_4g_{24} & l_4(1-g_{44})\end{vmatrix}\partial_3[g(p)p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3}p_4^{s_4-1}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{31} & -l_1g_{41}\\ -l_3g_{13} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{14} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}\partial_2[g(p)p_1^{s_1-1}p_2^{s_2}p_3^{s_3-1}p_4^{s_4-1}]\\
&+\begin{vmatrix} l_2(1-g_{22}) & -l_2g_{32} & -l_2g_{42}\\ -l_3g_{23} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{24} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}\partial_1[g(p)p_1^{s_1}p_2^{s_2-1}p_3^{s_3-1}p_4^{s_4-1}]\\
&+\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{31} & -l_1g_{41}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{32} & -l_2g_{42}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{14} & -l_4g_{24} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}g(p)p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1}p_4^{s_4-1}.
\end{aligned}\tag{96}$$

Here we have defined

$$s_1=l_1(1-g_{11})-l_2g_{12}-l_3g_{13}-l_4g_{14},\tag{97a}$$

$$s_2=-l_1g_{21}+l_2(1-g_{22})-l_3g_{23}-l_4g_{24},\tag{97b}$$

$$s_3=-l_1g_{31}-l_2g_{32}+l_3(1-g_{33})-l_4g_{34},\tag{97c}$$

$$s_4=-l_1g_{41}-l_2g_{42}-l_3g_{43}+l_4(1-g_{44}),\tag{97d}$$

$$g(p)=\ln p_1+\ln p_2+\ln p_3+\ln p_4.\tag{98}$$

After a tedious calculation, we find

$$G_4(p)=p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1}p_4^{s_4-1}[A_4+g(p)B_4],\tag{99}$$

where A_4 and B_4 are given by

$$\begin{aligned}A_4={}&s_1s_2s_3+s_1s_2s_4+s_1s_3s_4+s_2s_3s_4-l_1(1-g_{11})(s_2s_3+s_2s_4+s_3s_4)-l_2(1-g_{22})(s_1s_3+s_1s_4+s_3s_4)-l_3(1-g_{33})(s_1s_2\\&+s_2s_4+s_1s_4)-l_4(1-g_{44})(s_1s_2+s_1s_3+s_2s_3)+\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{21}\\ -l_2g_{12} & l_2(1-g_{22})\end{vmatrix}(s_3+s_4)+\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{31}\\ -l_3g_{13} & l_3(1-g_{33})\end{vmatrix}(s_2+s_4)\\&+\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{41}\\ -l_4g_{14} & l_4(1-g_{44})\end{vmatrix}(s_2+s_3)+\begin{vmatrix}l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}(s_1+s_4)+\begin{vmatrix}l_2(1-g_{22}) & -l_2g_{42}\\ -l_4g_{24} & l_4(1-g_{44})\end{vmatrix}(s_1+s_3)\\&+\begin{vmatrix}l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}(s_1+s_2)-\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{31}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}\\&-\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{41}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{42}\\ -l_4g_{14} & -l_4g_{24} & l_4(1-g_{44})\end{vmatrix}-\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{31} & -l_1g_{41}\\ -l_3g_{13} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{14} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}\\&-\begin{vmatrix}l_2(1-g_{22}) & -l_2g_{32} & -l_2g_{42}\\ -l_3g_{23} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{24} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix},\end{aligned}\tag{100}$$

$$\begin{aligned}B_4={}&s_1s_2s_3s_4-l_1(1-g_{11})s_2s_3s_4-l_2(1-g_{22})s_1s_3s_4-l_3(1-g_{33})s_1s_2s_4-l_4(1-g_{44})s_1s_2s_3+\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{21}\\ -l_2g_{12} & l_2(1-g_{22})\end{vmatrix}s_3s_4\\&+\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{31}\\ -l_3g_{13} & l_3(1-g_{33})\end{vmatrix}s_2s_4+\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{41}\\ -l_4g_{14} & l_4(1-g_{44})\end{vmatrix}s_2s_3+\begin{vmatrix}l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}s_1s_4\\&+\begin{vmatrix}l_2(1-g_{22}) & -l_2g_{42}\\ -l_4g_{24} & l_4(1-g_{44})\end{vmatrix}s_1s_3+\begin{vmatrix}l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}s_1s_2-\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{31}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{32}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})\end{vmatrix}s_4\\&-\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{41}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{42}\\ -l_4g_{14} & -l_4g_{24} & l_4(1-g_{44})\end{vmatrix}s_3-\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{31} & -l_1g_{41}\\ -l_3g_{13} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{14} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}s_2\\&-\begin{vmatrix}l_2(1-g_{22}) & -l_2g_{32} & -l_2g_{42}\\ -l_3g_{23} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{24} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}s_1+\begin{vmatrix}l_1(1-g_{11}) & -l_1g_{21} & -l_1g_{31} & -l_1g_{41}\\ -l_2g_{12} & l_2(1-g_{22}) & -l_2g_{32} & -l_2g_{42}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33}) & -l_3g_{43}\\ -l_4g_{14} & -l_4g_{24} & -l_4g_{34} & l_4(1-g_{44})\end{vmatrix}.\end{aligned}\tag{101}$$

From comparing these with the expressions of A_3 and B_3, we find the following:

$$\begin{aligned}A_4={}&\begin{vmatrix}l_1(1-g_{11})-s_1 & -l_1g_{21} & -l_1g_{31}\\ -l_2g_{12} & l_2(1-g_{22})-s_2 & -l_2g_{32}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})-s_3\end{vmatrix}+\begin{vmatrix}l_1(1-g_{11})-s_1 & -l_1g_{21} & -l_1g_{41}\\ -l_2g_{12} & l_2(1-g_{22})-s_2 & -l_2g_{42}\\ -l_4g_{14} & -l_4g_{24} & l_4(1-g_{44})-s_4\end{vmatrix}\\&+\begin{vmatrix}l_1(1-g_{11})-s_1 & -l_1g_{31} & -l_1g_{41}\\ -l_3g_{13} & l_3(1-g_{33})-s_3 & -l_3g_{43}\\ -l_4g_{14} & -l_4g_{34} & l_4(1-g_{44})-s_4\end{vmatrix}+\begin{vmatrix}l_2(1-g_{22})-s_2 & -l_2g_{32} & -l_2g_{42}\\ -l_3g_{23} & l_3(1-g_{33})-s_3 & -l_3g_{43}\\ -l_4g_{24} & -l_4g_{34} & l_4(1-g_{44})-s_4\end{vmatrix},\end{aligned}\tag{102}$$

$$B_4=\begin{vmatrix}l_1(1-g_{11})-s_1 & -l_1g_{21} & -l_1g_{31} & -l_1g_{41}\\ -l_2g_{12} & l_2(1-g_{22})-s_2 & -l_2g_{32} & -l_2g_{42}\\ -l_3g_{13} & -l_3g_{23} & l_3(1-g_{33})-s_3 & -l_3g_{43}\\ -l_4g_{14} & -l_4g_{24} & -l_4g_{34} & l_4(1-g_{44})-s_4\end{vmatrix}=0,\tag{103}$$

where $s_1,\ s_2,\ s_3,\ s_4$ are defined by Eq. (97). Substituting Eq. (97) together with Eqs. (102) and (103) into Eq. (95), we obtain

$$\begin{aligned}
g_{l_1l_2l_3l_4}&=A_4\frac{1}{l_1!l_2!l_3!l_4!}\left(\frac{\partial^{l_1+l_2+l_3+l_4-4}}{\partial p_1^{l_1-1}\partial p_2^{l_2-1}\partial p_3^{l_3-1}\partial p_4^{l_4-1}}p_1^{s_1-1}p_2^{s_2-1}p_3^{s_3-1}p_4^{s_4-1}\right)_{p_1=p_2=p_3=p_4=1}\\
&=A_4\left(\frac{1}{l_1!}\frac{\partial^{l_1-1}}{\partial p_1^{l_1-1}}p_1^{s_1-1}\right)_{p_1=1}\left(\frac{1}{l_2!}\frac{\partial^{l_2-1}}{\partial p_2^{l_2-1}}p_2^{s_2-1}\right)_{p_2=1}\left(\frac{1}{l_3!}\frac{\partial^{l_3-1}}{\partial p_3^{l_3-1}}p_3^{s_3-1}\right)_{p_3=1}\left(\frac{1}{l_4!}\frac{\partial^{l_4-1}}{\partial p_4^{l_4-1}}p_4^{s_4-1}\right)_{p_4=1}\\
&=A_4\left(\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)]\right)\left(\frac{1}{l_2!}(s_2-1)(s_2-2)\cdots[s_2-(l_2-1)]\right)\\
&\quad\times\left(\frac{1}{l_3!}(s_3-1)(s_3-2)\cdots[s_3-(l_3-1)]\right)\left(\frac{1}{l_4!}(s_4-1)(s_4-2)\cdots[s_4-(l_4-1)]\right). \qquad (104)
\end{aligned}$$

Similarly we have the following:

$$g_{l_1l_2l_30}=\frac{A_3(l_1,l_2,l_3)}{l_1!l_2!l_3!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)](s_2-1)(s_2-2)\cdots[s_2-(l_2-1)](s_3-1)(s_3-2)\cdots[s_3-(l_3-1)], \qquad (105)$$

$$g_{l_1l_200}=\frac{A_2(l_1,l_2)}{l_1!l_2!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)](s_2-1)(s_2-2)\cdots[s_2-(l_2-1)], \qquad (106)$$

$$g_{l_1000}=\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)]. \qquad (107)$$

We can continue the above argument to the case of an arbitrary M, indefinitely. We would like to summarize the final result for an arbitrary M as follows: Let us denote by $G_M(p)=p_1^{s_1-1}p_2^{s_2-1}\cdots p_M^{s_M-1}[A_M+g(p)B_M]$, where A_M and B_M are defined by

$$\begin{aligned}
A_M&=\sum_{j=1}^{M}s_1\cdots s_j^*\cdots s_M-\sum_{j<k=1}^{M}l_j(1-g_{jj})s_1\cdots s_j^*\cdots s_k^*\cdots s_M\\
&\quad+\sum_{i<j<k=1}^{M}\begin{vmatrix} l_i(1-g_{ii}) & -l_ig_{ji}\\ -l_jg_{ij} & l_j(1-g_{jj})\end{vmatrix}s_1\cdots s_i^*\cdots s_j^*\cdots s_k^*\cdots s_M+\cdots\\
&\quad+(-1)^{M-1}\sum_{j=1}^{M}\begin{vmatrix} l_1(1-g_{11}) & -l_1g_{21} & \cdots & -l_1g_{M1}\\ -l_2g_{12} & l_2(1-g_{22}) & \cdots & -l_2g_{M2}\\ \vdots & \vdots & \ddots & \vdots\\ -l_Mg_{1M} & -l_Mg_{2M} & \cdots & l_M(1-g_{MM})\end{vmatrix}_{jj} \qquad (108a)
\end{aligned}$$

$$\begin{aligned}
&=\begin{vmatrix} l_1(1-g_{11})-s_1 & -l_1g_{21} & \cdots & -l_1g_{M-1,1}\\ -l_2g_{12} & l_2(1-g_{22})-s_2 & \cdots & -l_2g_{M-1,2}\\ \vdots & \vdots & \ddots & \vdots\\ -l_{M-1}g_{1,M-1} & -l_{M-1}g_{2,M-1} & \cdots & l_{M-1}(1-g_{M-1,M-1})-s_{M-1}\end{vmatrix}+\cdots\\
&\quad+\begin{vmatrix} l_2(1-g_{22})-s_2 & -l_2g_{32} & \cdots & -l_2g_{M,2}\\ -l_3g_{23} & l_3(1-g_{33})-s_3 & \cdots & -l_3g_{M,3}\\ \vdots & \vdots & \ddots & \vdots\\ -l_Mg_{2,M} & -l_Mg_{3,M} & \cdots & l_M(1-g_{M,M})-s_M\end{vmatrix}, \qquad (108b)
\end{aligned}$$

$$B_M = s_1 \cdots s_M - \sum_{j=1}^{M} l_j(1-g_{jj}) s_1 \cdots s_j^* \cdots s_M + \sum_{j<k=1}^{M} \begin{vmatrix} l_j(1-g_{jj}) & -l_j g_{kj} \\ -l_k g_{jk} & l_k(1-g_{kk}) \end{vmatrix} s_1 \cdots s_j^* \cdots s_k^* \cdots s_M$$

$$+ \cdots + (-1)^M \begin{vmatrix} l_1(1-g_{11}) & -l_1 g_{21} & \cdots & -l_1 g_{M1} \\ -l_2 g_{12} & l_2(1-g_{22}) & \cdots & -l_2 g_{M2} \\ \vdots & \vdots & \ddots & \vdots \\ -l_M g_{1M} & -l_M g_{2M} & \cdots & l_M(1-g_{MM}) \end{vmatrix} \tag{109a}$$

$$= \begin{vmatrix} l_1(1-g_{11})-s_1 & -l_1 g_{21} & \cdots & -l_1 g_{M1} \\ -l_2 g_{12} & l_2(1-g_{22})-s_2 & \cdots & -l_2 g_{M2} \\ \vdots & \vdots & \ddots & \vdots \\ -l_M g_{1M} & -l_M g_{2M} & \cdots & l_M(1-g_{MM})-s_M \end{vmatrix} = 0, \tag{109b}$$

respectively. s_j^* means the elimination of s_j and $|\ |_{jj}$ means the determinant with the elimination of the jth row and the jth column. The proof of Eq. (109b) that B_M always vanishes is straightforward using properties of determinants. Hence, we obtain

$$g_{l_1 \cdots l_M} = A_M \prod_{a=1}^{M} \frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{110}$$

where s_j's are defined by Eq. (61). And the lower coefficients are given by

$$g_{l_1 0 \cdots 0} = \frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)], \tag{111a}$$

$$g_{l_1 l_2 0 \cdots 0} = A_2(l_1,l_2) \prod_{a=1}^{2} \frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{111b}$$

$$\vdots$$

$$g_{l_1 \cdots l_{M-2} 00} = A_{M-2}(l_1, \ldots, l_{M-2}) \prod_{a=1}^{M-2} \frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{111m-2}$$

$$g_{l_1 \cdots l_{M-1} 0} = A_{M-1}(l_1, \ldots, l_{M-1}) \prod_{a=1}^{M-1} \frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{111m-1}$$

respectively, and so forth.

We would like to note the following: The above result of $M=1$ is exactly equivalent to the result first obtained by Sutherland[8,19] a long time ago. The results for the cases of $M=2$ and 3 are exactly equivalent to the results recently obtained by Isakov, Mashkevich, and Ouvry,[20] Mashkevich,[21] and Isakov and Mashkevich.[22] However, our results with the present method, represented in terms of the language of determinants, are all new and exact without any approximation up to an arbitrary M. And our results justify the expressions of Mashkevich[21] and of Isakov and Mashkevich,[22] where they obtained their results from explicitly expanding Wu's functional relations using computer algebra such as MATHEMATICA, and conjectured the analytic expressions of the coefficients for the higher terms by guessing from the lower terms of the coefficients. In this way, the method that we have presented above is powerful enough to evaluate nontrivial coefficients of the generalized Lagrange series expansions. This is the main advantage of this method.

VII. CLUSTER COEFFICIENTS FOR PHYSICAL SYSTEMS

In this section, we are going to discuss the thermodynamics of several physical systems of multispecies gases with MFES as well as pure FES in the language of the cluster expansions. We will show that the results presented in the preceding section provide the exact cluster coefficients for the cluster expansions in each physical system.

A. Thermodynamics of anyons in the lowest Landau level (LLL)

As a simplest case, let us first consider the thermodynamics of anyons in the lowest Landau level (LLL) of energy $\epsilon = \hbar\omega/2$ with $\omega = qB/mc$ denoted by $i=0$.[11,12] In this case, there exists only one species of anyons with statistics g. Therefore, $g_{ij}^{ab} \equiv g\,\delta_{ij}\delta_{ab}\delta_{i0}$ and $G_i^a = G_0 \equiv N_\phi = qBV/\hbar c$. This gives the following equation of state:

$$PV = k_B T G_0 \ln(1 + 1/w_0), \tag{112a}$$

$$N = G_0 n_0 = G_0 \frac{1}{w_0 + g}, \tag{112b}$$

where $w_0^g (1 + w_0)^{1-g} = e^{\beta(\epsilon - \mu)}$. This was first obtained by Dasnières de Veigy and Ouvry[14] and Wu.[12] Using the Sutherland transformation, $\zeta_0 = 1 + 1/w_0$, we find $\zeta_0^g - \zeta_0^{g-1} = e^{\beta(\mu - \epsilon)}$ and $PV = k_B T G_0 \ln \zeta_0$. When we define $z = e^{\beta\mu}$, we find $N/V = z(\partial/\partial z)(P/kT)$. Following the argument of Sutherland[8,19] and that in the preceding section of the $M = 1$ case, the cluster expansions are given by

$$\frac{PV}{k_B T} = G_0 \ln \zeta_0 = G_0 \sum_{l=1}^{\infty} c_l(g) z^l e^{-l\beta\epsilon}, \tag{113a}$$

$$\frac{N}{V} = \frac{G_0}{V} \sum_{l=1}^{\infty} l c_l(g) z^l e^{-l\beta\epsilon}, \tag{113b}$$

where we have used the Sutherland coefficients[8,19] and Eq. (68):

$$\begin{aligned} c_l(g) &= \frac{1}{l!}(1 - lg)(2 - lg) \cdots [(l-1) - lg] \\ &= \frac{(-1)^{l+1}}{lg} \frac{[lg]!}{l![l(g-1)]!}. \end{aligned} \tag{114}$$

This result coincides with that first obtained by Dasnières de Veigy and Ouvry.[4]

B. Thermodynamics of Laughlin's incompressible 1/*m* fluid

Let us next consider the system of two species of excitations, quasiholes (labeled by $-$) and quasielectrons (labeled by $+$) in Laughlin's incompressible $1/m$ fluid (m being odd). The existence of the two excitations dictates the nontrivial MFES.

Following the argument of Wu and collaborators[12] and others,[20–22] MFES parameters are given by

$$g_{++} = 2 - \frac{1}{m}, \quad g_{--} = \frac{1}{m}, \quad g_{+-} = -g_{-+} = \frac{1}{m} - 2. \tag{115}$$

The single excitation degeneracy in the thermodynamic limit is $G_+ = G_- = (1/m) N_\phi$. Hence, the densities $n_\pm = N_\pm / V$ are given by

$$n_\sigma = G_\sigma \frac{w_{-\sigma} + g_{-\sigma,-\sigma} - g_{\sigma,-\sigma}}{(w_+ + g_{++})(w_- + g_{--}) - g_{+-} g_{-+}}, \tag{116}$$

where w_σ $(\sigma = \pm)$ satisfy the functional equations

$$w_\sigma^{g_{\sigma\sigma}} (1 + w_\sigma)^{1 - g_{\sigma\sigma}} \left(\frac{w_{-\sigma}}{1 + w_{-\sigma}} \right)^{g_{-\sigma,\sigma}} = e^{\beta(\epsilon_\sigma - \mu_\sigma)}. \tag{117}$$

Let us define the Sutherland transformations, $\zeta_\sigma = 1 + 1/w_\sigma$ for $\sigma = \pm$. Substituting these into Eq. (117), we get

$$(\zeta_\sigma^{g_{\sigma,\sigma}} - \zeta_\sigma^{g_{\sigma,\sigma} - 1}) \zeta_{-\sigma}^{g_{-\sigma,\sigma}} = e^{\beta(\mu_\sigma - \epsilon_\sigma)} \equiv \alpha_\sigma, \tag{118}$$

where $\mu_+ + \mu_- = 0$ is usually assumed such that $z_+ z_- = 1$. And the pressure P, the total number N, and the difference M between the numbers of the species of the system are given by

$$\frac{PV}{k_B T} = G_+ (\ln \zeta_+ + \ln \zeta_-), \tag{119a}$$

$$\frac{N}{V} = G_+ (n_+ + n_-), \tag{119b}$$

$$\frac{M}{V} = G_+ (n_+ - n_-), \tag{119c}$$

$$n_\pm = G_\pm z_\pm \frac{\partial}{\partial z_\pm} \ln \zeta_\pm, \tag{119d}$$

respectively. Generalizing the method of Sutherland,[8,19] let us define the expansions

$$\begin{aligned} \ln \zeta_\pm &= {\sum_{l_1, l_2 = 0}^{\infty}}' \; c_{l_1 l_2}^{\pm} \alpha_+^{l_1} \alpha_-^{l_2} \\ &= {\sum_{l_1, l_2 = 0}^{\infty}}' \; c_{l_1 l_2}^{\pm} z_+^{l_1} z_-^{l_2} e^{-l_1 \beta \epsilon_+} e^{-l_2 \beta \epsilon_-}, \end{aligned} \tag{120}$$

where $'$ means that $l_1 = l_2 = 0$ is excluded from the summation and $c_{l_1 l_2}^{\pm}(\{g_{\mu\nu}\})$ stand for functions of all MFES parameters $\{g_{++}, g_{+-}, g_{-+}, g_{--}\}$, and we have denoted $c_{l_1 l_2}^{\pm}(\{g_{\mu\nu}\})$ by $c_{l_1 l_2}^{\pm}$ for the sake of simplicity if there is no confusion. P and the densities $n_\pm$ are represented by

$$\frac{PV}{k_B T} = G_+ {\sum_{l_1, l_2 = 0}^{\infty}}' \; c_{l_1 l_2} \alpha_+^{l_1} \alpha_-^{l_2}, c_{l_1 l_2} = c_{l_1 l_2}^{+} + c_{l_1 l_2}^{-}, \tag{121a}$$

$$n_+ = {\sum_{l_1, l_2 = 0}^{\infty}}' \; l_1 c_{l_1 l_2}^{+} \alpha_+^{l_1} \alpha_-^{l_2}, \tag{121b}$$

$$n_- = {\sum_{l_1, l_2 = 0}^{\infty}}' \; l_2 c_{l_1 l_2}^{-} \alpha_+^{l_1} \alpha_-^{l_2}. \tag{121c}$$

Thus, once the coefficients $c_{l_1 l_2}^{\pm}$ are obtained, so is the equation of state of the system. Let us find the coefficients $c_{l_1 l_2}^{\pm}$. From Eq. (118) we find

$$\zeta_+ = 1 + \alpha_+ \zeta_+^{1 - g_{++}} \zeta_-^{-g_{-+}}, \tag{122a}$$

$$\zeta_- = 1 + \alpha_- \zeta_-^{1 - g_{--}} \zeta_+^{-g_{+-}}. \tag{122b}$$

Thus, we can use the result [see Eq. (78)] for the case of $M = 2$ in the preceding section. Let us define

$$s_1 = l_1(1 - g_{++}) - l_2 g_{+-}, \quad s_2 = -l_1 g_{-+} + l_2(1 - g_{--}). \tag{123}$$

Then we obtain $c_{l_1 l_2}^{\pm}$ and $c_{l_1 l_2} = c_{l_1 l_2}^{+} + c_{l_1 l_2}^{-}$ as

$$c^{+}_{l_1l_2}=-l_1g_{-+}\prod_{a=1}^{2}\frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{124a}$$

$$c^{-}_{l_1l_2}=-l_2g_{+-}\prod_{a=1}^{2}\frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{124b}$$

$$c_{l_1l_2}=c^{+}_{l_1l_2}+c^{-}_{l_1l_2}=-(l_1g_{-+}+l_2g_{+-})\prod_{a=1}^{2}\frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{124c}$$

respectively. And the lower coefficients are given by

$$c_{l_10}=g_{l_1}(s_1), \tag{125a}$$

$$c_{0l_2}=g_{l_2}(s_2), \tag{125b}$$

where $g_l(s)$ are the Sutherland coefficients of Eq. (111a). The first few terms of this result were obtained by Mashkevich [see Eqs. (13)–(16) in Ref. 21] and Isakov and Mashkevich [see Eqs. (2.13)–(2.20) in Ref. 22], where higher terms were guessed. Here we note that when $g_{+-}=g_{-+}=0$, the $c_{l_1l_2}$ is separated as $c_{l_1l_2}=c_{l_1}(g_{++})\delta_{l_1,0}+c_{l_2}(g_{--})\delta_{l_2,0}$, where $c_l(g)$ are the Sutherland coefficients of Eq. (114), since Eq. (118) is decoupled into the two equations, $\zeta_\sigma^{g_{\sigma,\sigma}}-\zeta_\sigma^{g_{\sigma,\sigma}-1}=\alpha_\sigma$ for $\sigma=\pm$.

C. Ideal multispecies quasiparticle gases with MFES

As was discussed recently by Nayak and Wilczek,[17] Isakov, Arovas, Myrheim, and Polychronakos,[18] and Iguchi,[19] the QSM formulation in the momentum representation allows us to evaluate the equation of state for an ideal gas with pure FES in arbitrary dimensions and obtain all the exact cluster coefficients in the cluster expansions as well as the virial coefficients.[18,19] We now show that this is also true for the system of a multispecies ideal gas with MFES. For this case, let us assume that $g^{ab}_{ij}=g_{ab}\delta_{ij}$, which defines $\zeta^a_i=\zeta_a(\mathbf{p})$. Then, we have Eqs. (21) and (22), which are represented by using a good quantum number $\mathbf{p}$. If two species of $a=\pm$ such as spin and charge excitations are taken into account, then we can use the above result of Eq. (124). Assume the particle energy $\epsilon_\pm(\mathbf{p})=\mathbf{p}^2/2m_\pm$ and take $\epsilon=\mathbf{p}^2/2m_+$ such that $\epsilon_-(\mathbf{p})=\tau\epsilon$ with $\tau=m_+/m_-$. Then, the density of states is given by $N_D(\epsilon)=(m_+/2\pi\hbar^2)^{D/2}[1/\Gamma(D/2)]\epsilon^{(D-2)/2}$.[19] Using $\alpha_\pm(\mathbf{p})=\exp\{\beta[\mu_a-\epsilon_\pm(\mathbf{p})]\}$ together with the expansions of $\ln\zeta_a(\mathbf{p})$ [Eq. (121)], we find

$$\frac{P}{k_BT}=\frac{1}{\lambda^D}\sum_{l_1,l_2=0}^{\infty}{}' \; b_{l_1l_2}z_+^{l_1}z_-^{l_2}, \tag{126a}$$

$$\frac{N}{V}=\frac{1}{\lambda^D}\sum_{l_1,l_2=0}^{\infty}{}' \; (l_1+l_2)b_{l_1l_2}z_+^{l_1}z_-^{l_2}, \tag{126b}$$

$$\frac{M}{V}=\frac{1}{\lambda^D}\sum_{l_1,l_2=0}^{\infty}{}' \; (l_1-l_2)b_{l_1l_2}z_+^{l_1}z_-^{l_2}, \tag{126c}$$

$$b_{l_1l_2}=\frac{c_{l_1l_2}}{(l_1+\tau l_2)^{D/2}}=-\frac{l_1g_{-+}+l_2g_{+-}}{(l_1+\tau l_2)^{D/2}}\prod_{a=1}^{2}\frac{1}{l_a!}(s_a-1)(s_a-2)\cdots[s_a-(l_a-1)], \tag{127a}$$

$$b_{l_10}=\frac{1}{(l_1+\tau d_2)^{D/2}}\frac{1}{l_1!}(s_1-1)(s_1-2)\cdots[s_1-(l_1-1)], \tag{127b}$$

$$b_{0l_2}=\frac{1}{(l_1+\tau d_2)^{D/2}}\frac{1}{l_2!}(s_2-1)(s_2-2)\cdots[s_2-(l_2-1)], \tag{127c}$$

where s_a's are defined by Eq. (123) and λ is the thermal length defined by $\lambda\equiv\sqrt{2\pi\hbar^2/m_+kT}$. These functions are regarded as generalized two-variable polylogarithms[27] and turn out to be generalized zeta functions such as Eisenstein series[28] if $\mu_\pm=0$ such that $z_+=z_-=1$.

This argument can be straightforwardly extended to the systems of M-species quasiparticles more than two species such that the energies are given by $\epsilon_a(\mathbf{p})=\mathbf{p}^2/2m_a$. Following the above argument, let us denote as $\epsilon=\mathbf{p}^2/2m_1$ such that $\epsilon_b(\mathbf{p})=\tau_b\epsilon$ with $\tau_b=m_b/m_1$. Suppose that the expansions are given by

$$\ln \zeta_a(\mathbf{p}) = \sum_{L=1}^{\infty} \sum_{l_1+l_2+\cdots+l_M=L} c^a_{l_1 l_2 \cdots l_M} \alpha_1^{l_1} \alpha_2^{l_2} \cdots \alpha_M^{l_M}, \tag{128}$$

where the coefficients $c^a_{l_1 l_2 \cdots l_M}$ are represented by

$$c^a_{l_1 l_2 \cdots l_M} = \left| \begin{array}{cccc} l_1(1-g_{11})-s_1 & -l_1 g_{21} & \cdots & -l_1 g_{M1} \\ -l_2 g_{12} & l_2(1-g_{22})-s_2 & \cdots & -l_2 g_{M2} \\ \vdots & \vdots & \ddots & \vdots \\ -l_M g_{1M} & -l_M g_{2M} & \cdots & l_M(1-g_{MM})-s_M \end{array} \right|_{aa} \prod_{b=1}^{M} \frac{1}{l_b!}(s_b-1)(s_b-2)\cdots[s_b-(l_b-1)]. \tag{129}$$

Here $|\ |_{aa}$ means that the ath row and the ath column are eliminated in the determinant and the s_j's are defined by

$$\begin{aligned} s_1 &= l_1(1-g_{11}) - l_2 g_{12} - \cdots - l_M g_{1M}, \\ s_2 &= -l_1 g_{21} + l_2(1-g_{22}) - \cdots - l_M g_{2M}, \\ &\vdots \\ s_M &= -l_1 g_{M1} - l_2 g_{M2} - \cdots + l_M(1-g_{MM}). \end{aligned} \tag{130}$$

Hence, we find the generalized cluster expansions as

$$\frac{P}{k_B T} = \frac{1}{\lambda^D} \sum_{L=1}^{\infty} \sum_{l_1+l_2+\cdots+l_M=L} b_{l_1 l_2 \cdots l_M} z_1^{l_1} z_2^{l_2} \cdots z_M^{l_M}, \tag{131a}$$

$$\frac{N}{V} = \frac{1}{\lambda^D} \sum_{L=1}^{\infty} \sum_{l_1+l_2+\cdots+l_M=L} L b_{l_1 l_2 \cdots l_M} z_1^{l_1} z_2^{l_2} \cdots z_M^{l_M}, \tag{131b}$$

$$b_{l_1 l_2 \cdots l_M} = \frac{\sum_{a=1}^{M} c^a_{l_1 l_2 \cdots l_M}}{(\tau_1 l_1 + \tau_2 l_2 + \cdots + \tau_M l_M)^{D/2}}. \tag{131c}$$

This corresponds to the results of Mashkevich[20] and of Isakov and Mashkevich[21] as a generalization to those for arbitrary dimensional systems with MFES. It is also regarded as a generalized many-variable polylogarithm[27] and a generalized many-variable zeta function when $z_1 = \cdots = z_M = 1$.[28]

VIII. CONCLUSION

In conclusion, we have presented a method that enables one to obtain the thermodynamics of an ideal gas of multispecies quasiparticles with MFES in arbitrary dimensions, using the generalization of the classical Lagrange theorem. This is thought of as a generalization of the method of Sutherland for the pure FES case in the CSM in one dimension[7] to the case of multispecies with MFES in arbitrary dimensions. The generalized Lagrange theorem and the Lagrange series expansions are of great importance in their own right, since they are very powerful when we applied them to the physical systems described before. By the aid of the theorem we have been able to obtain the exact cluster coefficients in the cluster expansions in all order without any approximation. This demonstrates the advantage of the present method. It will prove very interesting to further investigate the convergence of the generalized cluster expansions as well as to study the other systems of multispecies quasiparticles with MFES using this promising method.

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