HOW DOES LAGRANGE'S THEOREM SOLVE THERMODYNAMICS OF A MULTISPECIES QUASIPARTICLE GAS WITH MUTUAL FRACTIONAL EXCLUSION STATISTICS?

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Lagrange's theorem for one complex variable functions is generalized to that for multi-complex variable functions, and applied to obtaining thermodynamics and generalized cluster expansions for a multispecies quasiparticle gas with mutual fractional exclusion statistics.

In this Letter I would like to announce that classical Lagrange's theorem and Lagrange series expansion for functions of one complex variable are generalized to those for functions of many complex variables, and these generalized results enable one to obtain all exact cluster coefficients of the cluster expansion for a multispecies quasiparticle gas with mutual fractional exclusion statistics (MFES).2,3

In quantum statistical mechanics (QSM),4 it is well known that the grand partition function $Q$ for an $M$-species gas can be written as

$$Q = \sum_{N=0}^{\infty} \sum_{\sum_{a=1}^{M} N_a = N} Q_{N_1 N_2 \ldots N_M} z_1^{N_1} z_2^{N_2} \ldots z_M^{N_M}. \quad (1)$$

Here $Q_{N_1 N_2 \ldots N_M} (= Q_N)$ is the microcanonical partition function of $N$ particles and $z_a = \exp[\beta \mu_a]$, $\beta = 1/k_B T$ and $N_a = \sum_{i=1}^{N_a} N_i^a$ with $\mu_a$ the chemical potential of species $a$, $N_a$ ($N_a^a$) being the number of species $a$ (with a set of good quantum numbers, $i$), respectively. Thermodynamic potential $\Omega$ and the total number $N$ of the system are given from $Q$ as $\Omega = -PV = -k_B T \ln Q$, $N = \sum_a N_a = \sum_a z_a \partial Q / \partial z_a \ln Q$. Since $Q$ is expanded as $Q = 1 + Q_{10 \ldots 0 z_1} + \ldots$, $\ln Q$ must be expanded as $(1/V) \ln Q = b_{10 \ldots 0 z_1} + \ldots$. Hence, $\Omega$ and $N$ can be expanded in the form of the cluster expansion4 as

$$-\frac{\Omega}{V k_B T} = \frac{P}{k_B T} = \sum_{L=1}^{\infty} \sum_{\sum_{a=1}^{M} l_a = L} b_{l_1 l_2 \ldots l_M} z_1^{l_1} z_2^{l_2} \ldots z_M^{l_M}. \quad (2)$$

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\[
\frac{N}{V} = \sum_{L=1}^{\infty} \sum_{\sum h_i = L} L b_{1h_1} \cdots b_{Lh_L} e^{\frac{i}{y} z_1^1 \cdots z_M^M},
\]  
where \( b_{1h_1} \cdots b_{Lh_L} \) are called cluster coefficients.4

For the one-species case of \( M = 1 \), one can write as \( P/k_B T = F(z) = \sum_{l=1}^{\infty} b_l z^l \) and \( N/V = \frac{d}{dz} F(z) = \sum_{l=1}^{\infty} L b_l z^{l-1} \). Then, Lee and Yang’s theorem5 tells us that if there exists a singularity of \( F(z) \) on the positive real axis of a complex \( z \)-plane, then there is a phase transition in the system. Therefore, the convergence of the cluster expansion is exactly related to the existence of phase transition of the system. Thus, the explicit evaluation of the cluster coefficients is very important for the theory of phase transition. However, unless the single species case, much knowledge is still absent, since one usually does not know how to obtain the cluster coefficients for a multispecies case.

Recently, there have appeared some efforts in this direction.6-8 Isakov, Mashkevich and Ouvry,6 Mashkevich,7 and Isakov and Mashkevich8 have presented the analytic expressions of cluster coefficients in the cluster expansions for the systems with MFES. There they explicitly expanded the cluster expansions satisfying the Wu’s functional relations using a computer in order to obtain the expressions for the cluster coefficients up to several order and the higher terms were guessed to give conjectured analytic expressions. Although their results seem very plausible and reasonable, there has been no method which justifies the analytic expressions of the cluster coefficients. Therefore, much knowledge is still absent. In this Letter I will give a class of exact cluster coefficients is analytically obtained for the system of a multispecies quasiparticle gas with MFES2,3 without any approximation.

The original Haldane’s definition of FES2 is generalized to that of MFES with a set of good quantum numbers so that statistical interactions \( g_{ij}^{ab} \) are given by differential relations: \( \Delta D^a_i = -\sum_{b,j} g_{ij}^{ab} N_j^b \), where \( D^a_i \) is the dimension of the Hilbert space \( H^a_i \) of states of a single particle of species \( a \) and good quantum number \( i \), confined to a finite region of matter. \( D^a_i \) can change as particles are added, while keeping the boundary conditions and size of the condensed-matter region constant. It is given by the integrated form \( D^a_i = G_i^a - \sum_{j,b} g_{ij}^{ab} (N_j^b - \delta_{ij} \delta_{ab}) \), where \( G_i^a \) is a constant being interpreted as the number of available single particle states of species \( a \) with good quantum number \( i \) when no particle is presented in the system.

Following Wu’s QSM formulation for the MFES ideal gas,3 the microcanonical partition function \( Q_N \) is defined by

\[
Q_N = \sum_{N_i^a} W(\{N_i^a\}) e^{-\beta E(\{N_i^a\})},
\]  
\[
W(\{N_i^a\}) = \prod_{i,a} \frac{[D_i^a + N_i^a - 1]!}{N_i^a! [D_i^a - 1]!},
\]
where \( E(\{N_i^a\}) = \sum_{i,a} e_i^a N_i^a \). The most probable distribution of \( Q \) is given by taking the extremum condition: \( \frac{\delta}{\delta N_i^a} \{ \ln W(\{N_i^a\}) + \sum_{j,b} \beta (e_j^b - \mu_b) N_j^b \} = 0 \).
This yields the famous Wu’s distribution function $w_i^a$:

$$w_i^a = \frac{1}{n_i^a} - \sum_{j,b} g_{ij}^{ab} n_j^b,$$

$$(1 + w_i^a) \prod_{j,b} \left( \frac{w_j^b}{1 + w_j^b} \right)^{b_{ji}} = e^{\beta (\varepsilon_i^a - \mu_a)},$$

where $n_i^a = N_i^a / D_i^a$ and $g_{ij}^{ab} = g_{ij}^{ab} / G_i^a$. Now, thermodynamic potential $\Omega$ and the total number $N$ are written as

$$-\frac{\Omega}{V k_B T} = \frac{P}{k_B T} = \frac{1}{V} \sum_{i,a} G_i^a \ln \left( \frac{1 + w_i^a}{w_i^a} \right),$$

$$N = \frac{1}{V} \sum_{i,a} G_i^a n_i^a = \sum_{i,a} z_i^a \frac{\partial}{\partial z_i^a} \left( \frac{P}{k_B T} \right),$$

respectively. This is valid for all cases with different species and a set of good quantum numbers.

Let us define the following transformation: $\zeta_i^a = 1 + 1 / w_i^a$, which is a generalization of the transformation first adapted by Sutherland $^9$ for the Calogero–Sutherland model (CSM) $^9,10$ and $w_i^a = 1 / (\zeta_i^a - 1)$. Substituting this, Eqs. (7)–(9) are converted into the following:

$$\left[(\zeta_i^a)^{a_{ji}^b} - (\zeta_i^a)^{\delta_{ji}^a - 1}\right] \prod_{j,b(\neq i,a)} (\zeta_j^b)^{\beta_{ji}} = e^{\beta (\mu_a - \varepsilon_i^a)} \equiv \alpha_i^a,$$

$$\frac{P}{k_B T} = \frac{1}{V} \sum_{i,a} G_i^a \ln \zeta_i^a,$$

$$N = \frac{1}{V} \sum_{i,a} G_i^a z_i^a \frac{\partial}{\partial z_i^a} \ln \zeta_i^a.$$

Now the problem becomes how to obtain $\ln \zeta_i^a$ as a power series of $\alpha_i^a$. Suppose that $\ln \zeta_i^a$ is expanded in the following form:

$$\ln \zeta_i^a = \sum_{\{l_i^a\} \neq 0} c_i^a[\{l_i^a\}] \prod_{j,b} (\alpha_j^b)^{l_j^b},$$

where $c_i^a[\{l_i^a\}]$ stands for $c_i^a[\{l_i^a\}] (g_{ij}^{ab})$ that is a function of $\{l_i^a\}$ meaning the set of integers $\{l_i^1, l_i^2, \ldots \}$ and of $\{g_{ij}^{ab}\}$ meaning the set of all MFES parameters, and $^\prime$ means $l_1^1 = l_2^1 = \cdots = 0$ is excluded from the summation. Substitution of Eq. (13)
into Eqs. (11) and (12) yields the generalized cluster expansions of the forms of Eqs. (2) and (3) such as

\[
\frac{P}{k_B T} = \sum_{L=1}^{\infty} \sum_{i,a} b(|l_i^a|) \prod_{j,b} \frac{z_b^j}{z_b^j},
\]

(14)

\[
\frac{N}{V} = \sum_{L=1}^{\infty} \sum_{i,a} L b(|l_i^a|) \prod_{j,b} \frac{z_b^j}{z_b^j},
\]

(15)

\[
b(|l_i^a|) = \frac{1}{V} \sum_{\bar{i},\bar{a}} G_{\bar{i},\bar{a}} n_{\bar{i},\bar{a}} |e^{-\beta e^a_i}|.
\]

(16)

In this way, once \(c_i^a(|l_i^a|)\) is obtained from the set of functional equations among other species with MFES \(\text{Eq. (10)}\), then the problem can be solved. This method is thought of as a generalization of the method first adapted by Sutherland\(^9\) for the CSL\(^1\) in one dimension where only one variable \(\alpha\) appears for the pure FES to that for the system with MFES where many variables \(\alpha_i^a\) appear.

For example, in the case of the system of \(M\)-species ideal gas with MFES of \(g_{ij}^{ab} = g_{ab} \delta_{ij}\), one can define as \(\zeta_i^a = \zeta_i(p)\) and the particle energy \(c_i^a = c_a(p) = p^2/2m_a\) for \(a = 1, \ldots, M\). Then, Eqs. (10)–(12) are represented using good quantum number \(p\) as

\[
(\zeta_i(p) - \zeta_i(p)) \prod_{b / a} \zeta_b(p) \prod_{a} e^{\beta (\mu_a - \epsilon_a(p))} \equiv \alpha_a(p),
\]

(17)

\[
\frac{P}{k_B T} = \frac{1}{V} \sum_{p} \ln \zeta_a(p),
\]

(18)

\[
\frac{N}{V} = \sum_{a} \frac{N_a}{V} = \sum_{a} \frac{z_a}{V} \frac{\partial}{\partial z_a} \frac{1}{V} \sum_{p} \ln \zeta_a(p).
\]

(19)

Therefore, I obtain the cluster expansions of Eqs. (2) and (3) with

\[
b(|l_i^a|) = \frac{1}{V} \sum_{\bar{i},\bar{a}} c_{\bar{i},\bar{a}} n_{\bar{i},\bar{a}} e^{-\beta \epsilon_{\bar{a}}(p)}.
\]

(20)

Let us explicitly obtain the coefficients \(c_{\bar{i},\bar{a}}(|l_{\bar{i}}|)\). For this purpose I first present a generalization of Lagrange's theorem for Lagrange series for one complex variable functions\(^1\) to that for many complex variable functions.\(^1\) Denote by \(z = (z_1, z_2, \ldots, z_M)\) the set of complex variables \(z_1, z_2, \ldots, z_M\). This \(z\) defines an \(M\)-torus, \(C^M\). Define the \(M\)-disk of \(C^M = C_1 \times C_2 \times \cdots \times C_M\) such that \(|z_a - c_a| < r_a\), \(a = 1, \ldots, M\).\(^1\) Denote an analytic function of many complex variables by \(f(z)\), which is defined over a domain \(D \subset C^M\). Now one has the generalized Cauchy theorem\(^1\):

\[
f(z) = \left(\frac{1}{2\pi i}\right)^M \int_{C^M} \frac{f(\zeta)d\zeta_1 d\zeta_2 \cdots d\zeta_M}{(\zeta_1 - z_1)(\zeta_2 - z_2) \cdots (\zeta_M - z_M)}.
\]

(21)
Then this provides Taylor expansion of $f(z)$:

$$f(z) = \sum_{L=1}^{\infty} \sum_{l_1+l_2+\cdots+l_M=L} f_{l_1 l_2 \cdots l_M} z_1^{l_1} z_2^{l_2} \cdots z_M^{l_M} ,$$

(22)

$$f_{l_1 l_2 \cdots l_M} = \left( \frac{1}{2\pi i} \right)^M \int_{C_M} \frac{f(\zeta) d\zeta_1 d\zeta_2 \cdots d\zeta_M}{\zeta_1^{l_1+1} \zeta_2^{l_2+1} \cdots \zeta_M^{l_M+1}}$$

$$= \frac{1}{l_1! l_2! \cdots l_M!} \frac{g^{l_1 + l_2 + \cdots + l_M}}{\partial z_1^{l_1} \partial z_2^{l_2} \cdots \partial z_M^{l_M}} f(0,0,\ldots,0) .$$

(23)

Next, let us define a set of analytic functions of $M$ complex variables $F_a(z)$ for $a = 1,\ldots,M$, which are defined over a domain $D \subset \mathbb{C}^M$. Assume that $F_a(\zeta) = x_a(a = 1,\ldots,M)$ are satisfied at $z$ and give simple roots for each variable of $z_a$. Suppose now that $|F_a(\zeta)| \geq |x_a|$. If this is true as a generalized Rouche’s theorem, then Eq. (21) can be generalized to

$$g(z) = \left( \frac{1}{2\pi i} \right)^M \int_{C_M} g(\zeta) \left| \frac{\partial F_1(\zeta) F_2(\zeta) \cdots F_M(\zeta)}{\partial (\zeta_1, \zeta_2, \ldots, \zeta_M)} \right| (F_1(\zeta) - x_1)(F_2(\zeta) - x_2) \cdots (F_M(\zeta) - x_M)$$

$$\times d\zeta_1 d\zeta_2 \cdots d\zeta_M .$$

(24)

Let us use this to generalize Lagrange’s theorem for inverting the functions. Suppose that $F_a(\zeta) = (\zeta_a - p_a)/\phi_a(\zeta) (a = 1,\ldots,M)$ such that the Rouche’s condition is satisfied (i.e., $|x_a \phi_a| \leq |\zeta_a - p_a|, a = 1,\ldots,M$), where $p_a$ are all constants. Here the set of equations $F_a(\zeta) = x_a(a = 1,\ldots,M)$ gives

$$\zeta_a = p_a + x_a \phi_a(\zeta) (a = 1,\ldots,M) ,$$

(25)

which defines $x_a$ as functions of $\zeta$ such that $x_a = x_a(\zeta)(a = 1,\ldots,M)$. Let us invert as $\zeta_a = x_a(x)$. In this case Eq. (24) turns out to be

$$g(z) = \left( \frac{1}{2\pi i} \right)^M \int_{C_M} g(\zeta) |J_M| d\zeta_1 d\zeta_2 \cdots d\zeta_M$$

$$\times |J_M| = \det \left| \frac{\partial \phi_a(\zeta_j)}{\partial \zeta_j} \right| .$$

Using $1/(\zeta_a - x_a \phi_a) = \sum_{a=0}^{\infty} \phi_a(\zeta_a) / (\zeta_a - p_a)^{a+1}$ and expanding both numerator and denominator as a power series of $x_a(a = 1,\ldots,M)$, I obtain

$$g(z) = \sum_{L=1}^{\infty} \sum_{l_1+l_2+\cdots+l_M=L} g_{l_1 l_2 \cdots l_M} z_1^{l_1} z_2^{l_2} \cdots z_M^{l_M} ,$$

(27)

$$g_{l_1 l_2 \cdots l_M} = \frac{1}{l_1! l_2! \cdots l_M!} \frac{g^{l_1 + l_2 + \cdots + l_M}}{\partial p_1^{l_1-1} \partial p_2^{l_2-1} \cdots \partial p_M^{l_M-1}} G_M(p) .$$

(28)
Here $G_M(p) = G_M(p_1, p_2, \ldots, p_M)$ is defined by

$$G_M(p) = \partial_1 \cdots \partial_M \left[ g(p) \phi_1(p) \phi_2(p) \cdots \phi_M(p) \right]$$

and

$$= \sum_{j=1}^{M} l_j \partial_1 \cdots \partial_j^* \cdots \partial_M \left[ g(p) \phi_1(p) \phi_2(p) \cdots \phi_M(p) \partial_j \partial_j^* \cdots \partial_j^* \cdots \partial_M \right]$$

$$= \sum_{j<k=1}^{M} l_j l_k \partial_1 \cdots \partial_j^* \cdots \partial_k^* \cdots \partial_M$$

$$= \sum_{j=1}^{M} \left[ g(p) \phi_1(p) \phi_2(p) \cdots \phi_j(p) \phi_{j+1}(p) \cdots \phi_M(p) \partial_j \partial_j^* \cdots \partial_j^* \cdots \partial_M \right]$$

$$= \sum_{j=k=1}^{M} (-1)^M l_j \cdots l_M \left[ g(p) \phi_1(p) \phi_2(p) \cdots \phi_j(p) \phi_{j+1}(p) \cdots \phi_M(p) \partial_j \partial_j^* \cdots \partial_j^* \cdots \partial_M \right]$$

where $\partial_j = \frac{\partial}{\partial p_j}$ and $\partial_j^*$ means elimination of $\partial_j$. Hence, this is a generalization of Lagrange series for one complex variable functions to that for $M$ complex variable functions.

Let us apply the above formula to obtain the cluster coefficients. From Eq. (17), I find $\zeta_a = 1 + \alpha_a \phi_a(\zeta)$ where $\phi_a(\zeta) = \zeta_a^{1-\gamma a} \prod_{b \neq a} \zeta_b^{\gamma a}$. Since this has the form of Eq. (25) with $p_a = 1$ and $x_a = \alpha_a (a = 1, \ldots, M)$, the Rouché's condition is satisfied in our problem. Therefore, I use the above generalized Lagrange's theorem. To obtain the exact cluster coefficients, I define $g(z) = \sum_{n=1}^{\infty} \ln \zeta_n$, denoting $c_{1,2,\ldots,M}$ for our problem. Substitution of this into Eqs. (28)–(30) provides the exact cluster coefficients for all order.

I give some examples here. For $M = 1$, since the Lagrange series is given by

$$\ln(1/g) = \sum_{i=1}^{\infty} c_i \alpha_i^i$$

I find

$$c_i = \frac{1}{l!} \frac{\partial^{l-1}}{\partial p^{l-1}} G_1(p) = \frac{1}{l!} \frac{\partial^{l-1}}{\partial p^{l-1}} \left\{ \frac{\partial}{\partial p} (\ln p \phi(p)^l) - \ln p \frac{\partial}{\partial p} \phi(p)^l \right\} = \frac{1}{l!} \frac{\partial^{l-1}}{\partial p^{l-1}} (p^{-1} \phi(p)^l),$$

which gives with $s = l(1-g)$

$$c_i(s) = \frac{1}{l!} \left( \frac{\partial^{l-1}}{\partial p^{l-1}} P^{s-1} \right)_{p=1} = \frac{(-1)^{l+1}}{l!} \frac{[lg]}{l![(l-1)]!}.$$
This was first obtained by Sutherland using another method of inverting a series expansion a long time ago\textsuperscript{9,13} and recently reproved by the author\textsuperscript{14} using the Lagrange series expansion. For \( M = 2 \), define \( g(\zeta) = \ln \zeta_1 + \ln \zeta_2 = \sum_{i=1}^{\infty} a_i \zeta_i^{a_i} + \sum_{i=1}^{\infty} b_i \zeta_i^{b_i} \). Now, \( G_2(p) = \partial_1 \partial_2 (g(p) \phi_1(p) \phi_2(p)) - \partial_2 (g(p) \phi_1(p)) \partial_1 \phi_2(p) = \ln(g(p)) \phi_1(p) \phi_2(p) - m \partial_1 (g(p) \phi_1(p)) \partial_2 \phi_2(p) + \ln(g(p)) \phi_1(p) \phi_2(p) - m \partial_1 \phi_1(p) \phi_2(p) \). Since \( g(p) = \ln p_1 + \ln p_2 \) and \( p_1 = p_2 = 1 \), I have \( \partial_1 g(p) = 1/p_1 \), \( \partial_2 g(p) = 1/p_2 \), and \( \partial_1 \partial_2 g(p) = 0 \). Therefore, substituting these and \( \partial_1 \phi_1(p) = \partial_2 \phi_2(p) = 0 \) into \( G_2(p) \) yields \( G_2(p) = A_2 p_1^{-1} p_2^{-1} \), where \( A_2 = -(l g_{21} + m g_{12}) \), \( s_1 = l(1 - g_{11}) - m g_{12} \), and \( s_2 = -l g_{21} + m(1 - g_{22}) \). Hence, by substitution I obtain

\[
 c_m = \frac{1}{m!} \left( \frac{\partial^{l+m-n-2} G_2(p)}{\partial p_1^{l-1} \partial p_2^{m-1}} \right) A_2 \left( \frac{1}{m!} \frac{\partial^{l+m-n} G_2(p)}{\partial p_1^{l-1} p_1^{m-1}} \right) p_{l+1} \left( \frac{1}{m!} \frac{\partial^{l+m-n} G_2(p)}{\partial p_2^{l-1} p_2^{m-1}} \right) p_{m+1} .
\]

(33)

And \( c_0 = c_1(s_1) \) and \( c_m = c_m(s_2) \), where \( c_i(s) \) is the Sutherland coefficients given by Eq. (32). Here I note that when \( g_{21} = g_{12} = 0 \), \( c_m \) is separated as \( c_m = c_i(g_{11}) M_{m,0} + c_m(g_{22}) M_{i,0} \), since Eq. (17) is decoupled to two equations, \( \zeta_1^{s_1} - \zeta_2^{s_2} = a \) for \( a = 1, 2 \). This result was first obtained by Isakov, Mashkevich and Ouvry\textsuperscript{6}.

For \( M = 3 \), do the same thing for obtaining \( G_3(p) \) using Eq. (31) and substitute this into Eqs. (28)-(30). After a bit tedious calculation, I obtain

\[
 c_{mn} = \frac{1}{m! m!} \left( \frac{\partial^{l+m+n-3} G_3(p)}{\partial p_1^{l-1} \partial p_2^{m-1} \partial p_3^{n-1}} \right) A_3 \left( \frac{1}{m!} \frac{\partial^{l+m+n} G_3(p)}{\partial p_1^{l-1} p_1^{m-1} p_2^{n-1}} \right) p_{l+1} \left( \frac{1}{m!} \frac{\partial^{l+m+n} G_3(p)}{\partial p_2^{l-1} p_2^{m-1} p_3^{n-1}} \right) p_{m+1} \left( \frac{1}{m!} \frac{\partial^{l+m+n} G_3(p)}{\partial p_3^{l-1} p_3^{m-1} p_3^{n-1}} \right) p_{n+1} .
\]

(34)

where I have defined as \( s_1 = l(1 - g_{11}) - m g_{12} - n g_{13} \), \( s_2 = -l g_{21} + m(1 - g_{22}) - n g_{23} \), and \( s_3 = -l g_{31} - m g_{32} + n(1 - g_{33}) \) and \( A_3 \) is given by

\[
 A_3 = \begin{vmatrix}
 m(1 - g_{22}) - s_2 & -m g_{32} & -n g_{32} \\
 -n g_{23} & n(1 - g_{33}) - s_3 & l(1 - g_{31}) - s_1 \\
 -m g_{12} & m(1 - g_{22}) - s_2 & \end{vmatrix}
 = l^2 g_{21} g_{31} + l m g_{21} g_{32} + l n g_{21} g_{32} + m^2 g_{32} g_{12} + m n g_{32} g_{13}
 + l n g_{32} g_{12} + m n g_{12} g_{31} + n^2 g_{12} g_{31} .
\]

(35)

And again the relations such as \( c_{m0} = c_m(s_1, s_2) \) and \( c_{00} = c_i(s_1) \) hold. This corresponds to the results of Mashkevich [see Eqs. (13)-(16) in Ref. 7] and Isakov and Mashkevich [see Eqs. (2.13)-(2.20) in Ref. 8] for special models with MFES. There first few terms of the above results were analytically obtained by explicitly
expanding the series expansions while the higher terms were guessed to give general formulas. And the general formulas for an arbitrary number of species case were also all conjectured.

The above argument can be straightforwardly extended to the system of an $M$-species case. For example, consider the $M$-species quasiparticle gas with MFES. Denote the quasiparticle energies by $\epsilon = \frac{p^2}{2m_1}$ and $e_b(p) = \tau_b \epsilon$ with $\tau_b = m_b/m_1$, where the density of states is given by $N_D(\epsilon) = (m_1/2\pi \hbar^2)^{D/2}[1/\Gamma(D/2)]\epsilon^{(D-2)/2}$.14

Use this and the expansion, $g(z) = \sum_{a=1}^{M} \ln c_a(p) = \sum_{L=1}^{\infty} \sum_{L_1 + L_2 + \ldots + L_M = L} c_{L_1 L_2 \ldots L_M} \alpha_{L_1}^{1} \alpha_{L_2}^{1} \ldots \alpha_{L_M}^{1} \in$ Eqs. (18) and (19), where the coefficients $c_{L_1 L_2 \ldots L_M}$ are all given by the following formula:

$$c_{L_1 L_2 \ldots L_M} = \prod_{a=1}^{M} \left( \frac{1}{l_{a1}^{1} \partial p_{a1}^{-1}} \right) G_M(p) = A_M \prod_{a=1}^{M} \left( \frac{1}{l_{a1}^{1} \partial p_{a1}^{-1} p_{a}^{-1}} \right) p_{a}^{-1},$$

$$A_M = \begin{vmatrix}
 l_1(1 - g_{11}) - s_1 & -l_1 g_{12} & \ldots & -l_1 g_{1 M - 1,1} \\
 -l_2 g_{12} & l_2(1 - g_{22}) - s_2 & \ldots & -l_2 g_{2 M - 1,2} \\
 \vdots & \vdots & \ddots & \vdots \\
 -l_{M - 1} g_{1, M - 1} & -l_{M - 1} g_{2, M - 1} & \ldots & l_{M - 1}(1 - g_{M - 1, M - 1}) - s_{M - 1}
\end{vmatrix}$$

(36)

Here $s_i$'s are defined by $s_i = \sum_{j=1}^{M} l_j (\delta_{ij} - g_{ij})$ for $i = 1, 2, \ldots, M$. Thus, I finally find the cluster coefficients as

$$b_{1 \ldots M} = \frac{c_{1 \ldots M}}{(l_1 + \tau_2 l_2 + \ldots + \tau_M l_M)^{D/2}},$$

(38)

which becomes $b_{1 \ldots M} = c_{1 \ldots M}/L^{D/2}$ if $\tau_b = m_b/m_1 = 1$. Hence, the problem is solved. I note that the cluster expansions can be regarded as a generalized many variable polylogarithm and a generalized many variable zeta function when $z_1 = \cdots = z_n = 1$.15 And the above formula of Eqs. (36)-(38) would justify that of Mashkevich7 and Isakov and Mashkevich8 for the arbitrary species case.

In conclusion I have presented a method which enables one to obtain thermodynamics and all exact cluster expansions for a multispecies quasiparticle gas with MFES in arbitrary dimensions, using the generalized Lagrange’s theorem for many complex variable functions. It is very interesting to investigate convergence of the generalized cluster expansions as well as to study other systems of multispecies quasiparticles with MFES using this promising method.
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References

12. The detailed proof will be published elsewhere. K. Iguchi, unpublished.