QUANTUM STATISTICAL MECHANICS
AND THERMODYNAMICS OF A MULTISPECIES
QUASIPARTICLE GAS WITH MUTUAL
FRACTIONAL EXCLUSION STATISTICS

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Quantum statistical mechanics and thermodynamics of an ideal gas of multispecies quasiparticles with mutual fractional exclusion statistics are discussed. It is shown that thermodynamic potential and density of the system are analytically expressed in terms of the language of generalized cluster expansions, where the cluster coefficients are determined from Wu's functional relations of distribution functions for mutual fractional exclusion statistics.

1. The concept of spin-charge separation has attracted much interest of physicists for recent years. It has been believed to be very significant and responsible for the origins of high $T_c$ superconductivity, quantum Hall effect, anyon superconductivity, and Mott transition. Spin-charge separation is essentially realized by two types of excitations — spin and charge excitations — in the system. A microscopic derivation for this was first carried out on the Tomonaga–Luttinger model (TLM), the Calogero–Sutherland model (CSM), and the Haldane–Shastry model (HSM). The spin-charge separation was stemmed out as a consequence of the concept of Luttinger liquids in one dimension. Since spin and charge excitations are not independent of each other, this effect can be taken care of by mutual fractional exclusion statistics (MFES) between the two excitations. This situation was expected to be correct even for higher-dimensional systems. However, the system having spin-charge separation is not only the system but there are many other systems where MFES plays an important role. Therefore, it belongs to more broad systems of multispecies quasiparticles with MFES. Thus, this class of systems seems very important in the strongly correlated systems.

To understand thermal properties of such a system of multispecies quasiparticles with MFES, it is inevitably necessary to obtain the equation of state for the system. However, it has been extremely difficult to do so except for pure FES cases.
since MFES is given by a set of functional equations for the Wu’s distribution functions.\textsuperscript{11} Therefore, much knowledge is still absent, although some attempts have been made by several authors.\textsuperscript{19–21}

In this letter I present a method that enables one to resolve the above difficulty and to obtain the equation of state for the system of multispecies quasiparticles with MFES in arbitrary dimensions.

2. The original Haldane’s definition of FES is generalized to that of MFES with a set of good quantum numbers so that statistical interactions $\gamma_{ij}^{ab}$ are given by differential relations: $\Delta D_i^a = -\sum_{b,j} g_{ij}^{ab} \Delta N_j^b$, where $D_i^a$ is the dimension of the Hilbert space $H_i^a$ of states of a single particle of species $a$ and good quantum number $i$, confined to a finite region of matter, and $N_i^a$ the number of particles of species $a$ with a set of good quantum numbers $i$. $D_i^a$ can change as particles are added, while keeping the boundary conditions and size of the condensed-matter region constant.

By integration, it is also given as $D_i^a = G_i^a - \sum_{j,b} g_{ij}^{ab}(N_i^b - \delta_{ij}\delta_{ab})$, where $G_i^a$ is a constant being interpreted as the number of available single particle states of species $a$ with good quantum number $i$ when no particle is presented in the system.

3. Following Wu’s quantum statistical mechanics (QSM) formulation in the state representation,\textsuperscript{11,15} the grand partition function of the system is given by

$$Q = \sum_{N=0}^{\infty} \sum_{N_i^a, N_i^a} \prod_{i,a} \left( \frac{z_i^N}{z_i^{N_i^a}} \right) Q_N,$$

(1)

where $z_i$ is the fugacity defined by $z_i \equiv \exp[\beta\mu_i]$ with $\beta \equiv 1/k_B T$ ($k_B$ is the Boltzmann constant and $T$ the temperature) and $\mu_i$ the chemical potential of particles of species $a$, and $N$ the total number of particles defined by $N = \sum_{i,a} N_i^a$. Now, the microcanonical partition function $Q_N$ is defined by

$$Q_N = \sum_{\{N_i^a\}} W(\{N_i^a\}) e^{-\beta E(\{N_i^a\})},$$

(2)

$$W(\{N_i^a\}) = \prod_{i,a} \frac{D_i^a + N_i^a - 1}{N_i^a!!(D_i^a - 1)!!},$$

(3)

$$E(\{N_i^a\}) = \sum_{i,a} e_i^a N_i^a.$$  

(4)

Substituting Eqs. (2)–(4) into Eq. (1), the most probable distribution of $Q$ is given by taking the extremum condition:

$$\frac{\delta}{\delta N_i^a} \left\{ \ln W(\{N_i^a\}) + \sum_{j,b} \beta (e_{ij}^{ab} - \mu_b) N_j^b \right\} = 0.$$

This yields the famous Wu’s distribution function $w_i^a$:

$$w_i^a = \frac{1}{N_i^a} - \sum_{j,b} g_{ij}^{ab} \frac{N_j^b}{N_i^a},$$

(5)
\[(1 + w_i^a) \prod_{j,b} \left( \frac{u_{ij}^b}{1 + w_i^b} \right)^{g_i^{ab}} = e^{\beta(\epsilon_i^a - \mu_a)}, \tag{6}\]

where \(n_i^a \equiv N_i^a / D_i^a\) and \(g_i^{ab} \equiv g_i^{ab} G_i^b / G_i^a\). Now, thermodynamic potential \(\Omega\) and the total number \(N\) are obtained as

\[\Omega = -PV = -k_B T \sum_{i,a} G_i^a \ln \left( \frac{1 + w_i^a}{w_i^a} \right), \tag{7}\]

\[N = \sum_{i,a} G_i^a n_i^a = \sum_{i,a} z_i^a \frac{\partial}{\partial z_i^a} \left( \frac{PV}{k_B T} \right), \tag{8}\]

respectively, which are valid for all cases with different species and a set of good quantum numbers.

4. Main difficulties for calculating the thermodynamic potential lied on the fact that Eq. (6) represents a set of very complicated functional equations among other MFES. Therefore, when the MFES is taken into account, there has been no good method to explicitly obtain the Wu's distribution functions \(w_i^a\), except for the special cases recently considered by Mashkevich and Isakov and Mashkevich. I now present a method to resolve this problem.

Let us first define the following transformation:

\[\zeta_i^a = \frac{1 + w_i^a}{w_i^a}, \tag{9}\]

which is a generalization of the transformation first adapted by Sutherland and \(1/(\zeta^a - 1)\). Substituting Eq. (9) into Eq. (6), I obtain the following:

\[(\zeta_i^a)^{\alpha_i^a} - (\zeta_i^a)^{\alpha_i^a - 1} \prod_{j,b(\neq i,a)} \zeta_j^b = e^{\beta(\mu_a - \epsilon_i^a)} \equiv \alpha_i^a, \tag{10}\]

while Eqs. (7) and (8) are converted into

\[PV = k_B T \sum_{i,a} G_i^a \ln \zeta_i^a, \quad N = \sum_{i,a} G_i^a n_i^a \frac{\partial}{\partial z_i^a} \ln \zeta_i^a. \tag{11}\]

Defining \(\zeta_i^a \equiv \omega_i^a\) and substituting \(\zeta_i^a = \exp[\omega_i^a]\) into Eq. (10), it turns out to be

\[(e^{\alpha_i^a - \omega_i^a} - e^{(\alpha_i^a - 1)\omega_i^a}) \prod_{j,b(\neq i,a)} e^{\alpha_j^b \omega_j^b} = \alpha_i^a. \tag{12}\]

Equation (12) is solved in the following way: In this case of MFES, \(\omega_i^a\) does not only depend on \(\alpha_i^a\) but also depends on other \(\alpha_j^b\). So, let us first define

\[\omega_i^a \equiv \sum_{\{t_i^a\} = 0}^\infty \sum_{\{t_j^b\} = 0}^\infty \sum_{\{t_j^b\} = 0}^\infty c[\{t_i^a\}] \prod_{j,b} (\alpha_j^b)^{t_j^b} = \sum_{\{t_i^a\} = 0}^\infty \sum_{\{t_j^b\} = 0}^\infty \prod_{j,b} (z_i^a e^{-\beta \alpha_j^b}), \tag{13}\]
where $c_\alpha(l_\alpha^n)^2$ stands for $c_\alpha(l_\alpha^n)(g_{ab}^{\beta\gamma})$ that is a function of $\{l_\alpha^n\}$ meaning the set of integers $\{l_\alpha^n, l_\beta^n, \ldots\}$ and of $\{g_{ab}^{\beta\gamma}\}$ meaning the set of all MFES parameters, and $i'$ means $l_{\alpha'}^n = l_{\beta'}^n = \cdots = 0$ is excluded from the summation. Second, pick up a given set of $a$ and $i'$ to fix $\alpha^n_\alpha$ in the right hand side of Eq. (12), and expand the left hand side of Eq. (12) with respect to $\omega^n_\alpha$, so that it becomes an infinite power series of $\omega^n_\alpha$. Third, substitute the expansions Eq. (13) for all $\omega^n_\alpha$ into it so that it becomes an infinite power series of all $\alpha^n_\alpha$. Fourth, compare coefficients of both sides of the equation with respect to each power term of $\alpha^n_\alpha$, which provide an infinite set of relations for $c_\alpha(l_\alpha^n)^2$ for the given set of $a$ and $i'$, respectively. Since there are many such relations for $\omega^n_\alpha$ for other sets of $a$ and $i'$, do the same thing for these sets, respectively. One now finds many sets of relations for $c_\alpha(l_\alpha^n)^2$ for different sets of $a$ and $i'$. Finally, from these relations, determine the coefficients $c_\alpha(l_\alpha^n)^2$, successively. Then, all $\ln \zeta^n_\alpha$ are obtained such that the generalized cluster expansions are given by

$$\frac{P}{k_B T} = \sum_{L=1}^{\infty} \sum_{i,a} b[(l_i^n)] \prod_{j,b} z_b^i, \quad (14)$$

$$\frac{N}{V} = \sum_{L=1}^{\infty} \sum_{i,a} \left( \sum_{j,a} \right) b[(l_i^n)] \prod_{j,b} z_b^i, \quad (15)$$

where $b[(l_i^n)]$ are the generalized cluster coefficients.\textsuperscript{22}

In this way, the set of functional equations among other species with MFES [Eq. (6)] can be solved in principle. This method is thought of as a generalization of the method first adapted by Sutherland\textsuperscript{7} for the CSM in one dimension where only one variable $a$ appears for the pure FES to that for the system with MFES where many variables appear $\alpha^n_\alpha$. Let us now apply the above formulas to the systems of multiple species quasiparticles with MFES.

5. As a simplest case, let us first consider thermodynamics of anyons in the lowest Landau level (LLL) of energy $\epsilon = \hbar \omega/2$ with $\omega = qB/mc$ denoted by $i = 0$.\textsuperscript{11,13} In this case, there is only one species of anyons with statistics $g$. Therefore, $g^{\delta_{a,b}} = g^{\delta_{a,b}} \delta_{a,0}$ and $G^n_\alpha = G_0 \equiv N_0 = qBV/hc$. This gives $PV = k_B T G_0 \ln(1 + 1/w_0)$ and $N = G_0 n_0 = G_0/(w_0 + g)$, where $w_0 (1 + w_0)^{1-g} = e^{g(\xi - n)}$. This was obtained by Dasnieres de Veigy and Ouvry\textsuperscript{13} and Wu.\textsuperscript{11} Using the transformation, $\zeta_0 = 1 + 1/w_0$, I find $\zeta_0 - \zeta_0^{g-1} = e^{g(\mu - e)}$ and $PV = k_B T G_0 \ln \zeta_0$. When $z = e^{g\beta}$ is defined, I find $N/V = z(\partial / \partial z)(P/KT)$. Following the argument of Sutherland\textsuperscript{7,18} the cluster expansions are given by $PV = k_B T = G_0 \ln \zeta_0 = G_0 \sum_{l=1}^{\infty} c_l(g) z^l e^{-\beta \xi}$ and $N/V = (G_0/V) \sum_{l=1}^{\infty} l c_l(g) z^l e^{-\beta \xi}$, where I have used Sutherland expansion\textsuperscript{7,18}.

$$\ln \zeta_0 = \sum_{l=1}^{\infty} c_l(g) z^l e^{-\beta \xi}, \quad c_l(g) = \frac{(-1)^{l+1}}{l!} \frac{[l g]!}{l!(l(g - 1))!}, \quad (16)$$

This result coincides with that first obtained by Dasnieres de Veigy and Ouvry.\textsuperscript{13}
6. Let us next consider the system of two species of excitations, quasiholes (labeled by $-$) and quasi-electrons (labeled by $*$) in the Laughlin's incompressible 1/m fluid ($m$ being odd). The existence of the two excitations dictates nontrivial MFES. Following the argument of Wu and collaborators, the MFES parameters are given by $g_{++} = 2 - 1/m, g_{+-} = 1/m, g_{-+} = -g_{-} = (1/m) - 2$, and the single excitation degeneracy in the thermodynamic limit is $G_+ = G_- = (1/m)N_\phi$. Hence, the densities $n_\sigma = N_\sigma/V$ are given by

$$n_\sigma = \frac{G_\sigma w_\sigma + g_{-\sigma} - g_{\sigma} - g_{-\sigma}}{w_\sigma + g_{++} + g_{--}}$$

(17)

where $w_\sigma (\sigma = \pm)$ satisfy the functional equations

$$u^\sigma_{\bar{\sigma} \sigma} (1 + w_\sigma) - g_{\bar{\sigma} \sigma} \left( \frac{w_\sigma}{1 + w_\sigma} \right)^{g_{\bar{\sigma} \sigma} - g_{\sigma} - g_{-\sigma}} = e^{\beta (\epsilon_\sigma - \mu_\sigma)}.$$  

(18)

Let us define the transformations, $\zeta_\sigma = 1 + 1/w_\sigma$ for $\sigma = \pm$. Substituting into Eq. (18) yields

$$(c^\sigma_{\bar{\sigma} \sigma} - \zeta_\sigma^{g_{\bar{\sigma} \sigma} - 1})^{\zeta_\sigma^{g_{\bar{\sigma} \sigma} - g_{\sigma} - g_{-\sigma}}} = e^{\beta (\epsilon_\sigma - \mu_\sigma)} \equiv \alpha_\sigma,$$

(19)

where if $\mu_+ + \mu_- = 0$ then $z_+ z_- = 1$. And the pressure $P$, the total number $N$, and the difference $M$ between the numbers of the species of the system are given by $PV/k_B T = G_+ (\ln \zeta_+ + \ln \zeta_-), N/V = G_+ (n_+ + n_-), M/V = G_+ (n_+ - n_-)$, respectively, where $n_\pm = G_\pm z_\pm \frac{2}{G_\pm} \ln \zeta_\pm$. Generalizing the method of Sutherland, let us define the expansions:

$$\ln \zeta_\pm = \sum_{l,m=0}^{\infty} c^\pm_{lm} \frac{\alpha_+^l \alpha_-^m}{\alpha_+^l \alpha_-^m} e^{-12\epsilon_+ \epsilon_-} e^{-m\beta \epsilon_-},$$

(20)

where $'$ means that $l = m = 0$ is excluded from the summation and I have denoted $c^\pm_{lm}(\{g_{mn}\})$ by $c^\pm_{lm}$ for the sake of simplicity, which are functions of all the MFES parameters $\{g_{++}, g_{--}, g_{-+}, g_{+-}\}$. Then, the pressure $P$ and the densities $n_\pm$ are represented by

$$\frac{PV}{k_B T} = G_+ \sum_{l,m=0}^{\infty} c^+_{lm} \alpha_+^l \alpha_-^m, c^+_m = c^+_{lm} + c^-_{lm},$$

(21)

$$n_+ = G_+ \sum_{l,m=0}^{\infty} c^+_l \alpha_+^l \alpha_-^m, n_- = G_- \sum_{l,m=0}^{\infty} c^-_l \alpha_+^l \alpha_-^m.$$  

(22)

Thus, once the coefficients $c^\pm_{lm}$ are obtained, so is the equation of state of the system.

Let us find the coefficients $c^\pm_{lm}$. Define $\zeta_\pm = \omega_\pm$, and inversely $\zeta_\pm = \exp(\omega_\pm)$. Substitute these into Eq. (19), I find

$$(e^{\omega_+ - (g_{++} - 1)\omega_+} - e^{g_{-+} - \omega_+}) = e^{\omega_+ - \omega_+} = \alpha_+,$$

(23)

$$(e^{\omega_- - (g_{--} - 1)\omega_-} - e^{g_{-+} - \omega_-}) = e^{\omega_- - \omega_-} = \alpha_-.$$  

(24)
Consider first Eq. (23). Expand the left hand side with respect to $\omega_{\pm}$. Substitute the expansions of $\omega_{\pm}$ into them. Compare both sides for each coefficient of powers of $\alpha_\pm$ and determine the relations for $c^\pm_{im}$. Do the same thing for Eq. (24), once again. From these relations determine $c^{-}_{im}$. Then, I obtain all the coefficients $c^\pm_{im}$. Hence, the problem is solved$^{22}$ such that $c_{im} = c^+_{im} + c^-_{im}$. I present all coefficients$^{23}$:

$$c_0 = c_1(s_1), \quad c_{0m} = c_m(s_2),$$  

$$c_{im} = \frac{\lambda_s - m g_{s \pm} - m g_{s -}}{t! m!} \left( \frac{\partial^{l-m-2}}{\partial \alpha_{s}^{l-1} \partial \beta_{m-1}} \left( a^{s_1-1} b^{s_2-1} \right) \right)_{a=b=1},$$  

(25)

where $s_1 = l(1 - g_{s \pm}) - m g_{s -}$, $s_2 = -l g_{s -} + m(1 - g_{s -})$, $c_l(g)$ are the Sutherland coefficients of Eq. (16). First few terms of this result were first obtained by Mashkevich [see Eqs. (13)–(16) in Ref. 20] and Isakov and Mashkevich [see Eqs. (2.13)–(2.20) in Ref. 21] where higher terms were guessed. Here I note that when $g_{s +} = g_{s -} = 0$, $c_{im}$ is separated as $c_{im} = c_l(g_{s +}) \delta_{m,0} + c_m(g_{s -}) \delta_{l,0}$ where $c_l(g)$ are the Sutherland coefficients [Eq. (16)], since Eq. (19) is decoupled into two equations, $\xi^\sigma_{\alpha,\beta} - \xi^\sigma_{\alpha,\beta}^{-1} = \alpha_{\sigma}$ for $\sigma = \pm$.

7. As was discussed recently by Nayak and Wilczek,$^{16}$ Isakov, Arovas, Myrheim, and Polychronakos,$^{17}$ and the author,$^{18}$ the QSM formulation in the momentum representation allows us to evaluate the equation of state for an ideal gas with pure FES in arbitrary dimensions with obtaining all the exact cluster coefficients in the cluster expansions as well as virial coefficients.$^{17,18}$ I now show that this is also true for the system of multispecies ideal gas with MFES. For this case, let us assume that $g^{ij}_{a} = g_{ab} \delta_{ij}$, which defines $\xi_a^\sigma = \zeta_a(p)$. Then, Eqs. (11) and (12) are represented using good quantum number $p$ as

$$\left( \zeta_a(p)^{g_{a \sigma}} - \zeta_a(p)^{g_{a \sigma}^{-1}} \right) \prod_{b \neq a} \zeta_b(p)^{g_{ab}} = e^{\beta(\mu_a - \epsilon_a(p))} = \alpha_a(p),$$  

(26)

$$\frac{P}{k_B T} = \frac{1}{V} \sum_{a,p} \ln \zeta_a(p),$$  

(27)

$$\frac{N}{V} = \sum_{a} \frac{N_a}{V} = \sum_{a} 2a \partial_{\zeta_a} \frac{1}{V} \sum_{p} \ln \zeta_a(p),$$  

(28)

where $\zeta_a = \exp[\beta \mu_a]$.

8. If two species of $a = \pm$ such as spin and charge excitations are taken into account, then I can use the above result of Eq. (22). Assume the particle energy $\epsilon_{\pm}(p) = p^2/2m_{\pm}$ and take $\epsilon = p^2/2m_+$ such that $\epsilon(p) = \tau \epsilon$ with $\tau = m_+/m_-$. Then, the density of states is given by $N_D(\epsilon) = (m_+/2\pi \hbar^2)^{D/2} \left[ 1/T(D/2) \right] \epsilon^{(D-2)/2}$. Using $\alpha_{\pm}(p) = \exp[\beta(\mu_\pm - \epsilon_{\pm}(p))]$ together with the expansions of $\ln \zeta_{\pm}(p)$ [Eq. (20)], I find

$$\frac{P}{k_B T} = \frac{1}{\lambda^D} \sum_{l,m=0}^{\infty} b_{lm} \zeta_{\pm}^l \zeta_{-}^m,$$  

(29)
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\[ \frac{N}{V} \left( \frac{M}{V} \right) = \frac{1}{\lambda^D} \sum_{l,m=0}^{\infty} \{1 + (-1)^m\} b_{lm} z_+^l z_-^m, \quad (30) \]
\[ b_{lm} = \frac{e_{lm}}{(l + \tau m)^{D/2}}, \quad (31) \]

where \( \lambda \) is the thermal length defined by \( \lambda \equiv \sqrt{2 \pi \hbar^2 / m k_B T} \). These functions are regarded as generalized two-variable polylogarithms\(^{24}\) and turn out to generalized zeta functions such as Eisenstein series\(^{25}\) if \( \mu_+ = 0 \) such that \( z_+ = z_- = 1 \).

9. This argument can be straightforwardly extended to the systems of \( K \)-species quasiparticles more than two species such that energies are given by \( \epsilon_\alpha(p) = p^2 / 2m_\alpha \). Following the above argument taking \( \epsilon = p^2 / 2m_\alpha \), suppose that the expansions are given by \( \ln \zeta_\alpha(p) = \sum_{L=1}^{\infty} \sum_{m+n=0}^{L} c_L^{\alpha} \zeta_1 z_2 \cdots \zeta_K \), where the coefficients \( c_L^{\alpha} \) are assumed to be known. Hence, I find the generalized cluster expansions as

\[ \frac{P}{k_B T} = \frac{1}{\lambda^D} \sum_{L=1}^{\infty} \sum_{l+m+n=L} b_{lmn} z_1^l z_2^m \cdots z_K^n, \quad (32) \]
\[ \frac{N}{V} = \frac{1}{\lambda^D} \sum_{L=1}^{\infty} \sum_{l+m+n=L} L b_{lmn} z_1^l z_2^m \cdots z_K^n, \quad (33) \]
\[ b_{lmn} = \frac{\sum_{a=1}^{K} c_L^{\alpha} \zeta_{1} \cdots \zeta_{K} \zeta_{a}^{(\alpha)}}{(\tau_1 l + \tau_2 m + \cdots + \tau_K n)^{D/2}}. \quad (34) \]

This corresponds to the results of Mashkevich\(^{20}\) and Isakov and Mashkevich\(^{21}\) as a generalization to those for arbitrary dimensional systems with MFES. It is also regarded as a generalized many variable polylogarithm\(^{24}\) and a generalized zeta function when \( z_1 = \cdots = z_K = 1 \).

10. In conclusion I have presented a method which enables one to obtain thermodynamics of an ideal gas of multispecies quasiparticles with MFES in arbitrary dimensions. This is thought of as a generalization of the method of Sutherland for the pure FES case in the CSM in one dimension.\(^7\) It is very interesting to investigate convergence of the generalized cluster expansions as well as to study other systems of multispecies quasiparticles with MFES using this method.

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References

22. This problem can be thought of as a problem of inverting a set of two equations: 

\[
 f(x, y) = u, \quad g(x, y) = v \quad \text{such that} \quad x = \phi(u, v), \quad y = \psi(u, v). \]

In our case here, 

\[
 f(x, y) = (e^{ax} - e^{(a-1)x})e^{by}, \quad g(x, y) = (e^{cx} - e^{(c-1)x})e^{dx}, \]

where \(a, b, c, d\) are constants, and one would like to find the forms \(\phi(u, v)\) and \(\psi(u, v)\) explicitly. It is carried out as follows: Assume the forms as power series of \(u\) and \(v\) such that

\[
 \phi(u, v) = \sum_{m=1}^{\infty} p_m u^m v^n ,
\]

\[
 \psi(u, v) = \sum_{n=1}^{\infty} q_n u^n v^m.
\]
\[ \psi(u,v) = \sum_{m=1}^{\infty} q_m u^m. \]

Substitute them into the relations. And find all coefficients, \( p_m \) and \( q_m \). Thus, this method is a generalization of that of Sutherland for the CSM to the case of two variables. For the case with more species, it is generalized to the problems with many variables.

23. The proof will be published elsewhere.
