GENERALIZED SOMMERFELD THEORY: SPECIFIC HEAT OF A DEGENERATE g-ON GAS IN ANY DIMENSION AND THE GENERALIZED RIEMANN ZETA FUNCTION

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We generalize the Sommerfeld theory for a metal where the low temperature and high density expansions are known as the Sommerfeld expansions to that for a degenerate g-on gas — an ideal gas with fractional exclusion (i.e. Haldane–Wu) statistics of $0 \leq g \leq 1$ — in $D$ dimensions, using the quantum statistical mechanics formulation in the $D$-dimensional momentum representation. Using the generalized Sommerfeld expansions, the specific heat of the g-on gas is obtained herein, in terms of a generalized Riemann zeta function that is a natural extension of the Riemann zeta function. When $g > 0$, the specific heat of a g-on gas shows a linear $T$ dependence at low temperature as well as that of a metal, but the coefficient depends on statistics $g$ and dimensionality $D$ of the system. Therefore, we claim that this effect of statistics $g$ is testable by an experiment.

1. Introduction

We have witnessed that the concept of fractional exclusion statistics (FES) originated by Haldane's pioneering work in 1991 and the quantum statistical mechanics (QSM) formulation of an ideal gas with FES by Wu's seminal work in 1994 become a powerful tool in order to investigate the quantum statistical mechanical properties of the low-dimensional systems such as the fractional quantum Hall system and the Calogero–Sutherland model (CSM). Meantime, there have appeared many studies on the concept of FES for these low-dimensional systems. Especially, Daminides de Veigy and Ouvry have revealed a deep connection between the FES and the fractional quantum Hall system preceding one year before Wu's work, while later on, Isakov and Bernard and Wu have revealed the one between the FES and the CSM. And also, thermodynamics of these systems has been studied by many authors. On the other hand, higher-dimensional systems have not been studied by many authors except the work of Nayak and Wilczek and Isakov et al. and therefore much knowledge is still absent. The main reason for this seemed to lie on...

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the following facts: First, it has been believed that strictly speaking, the FES is realized only in one dimension.\textsuperscript{8,13,14} In the way that the Hilbert space in the lowest Landau level (LLL) of the fractional quantum Hall system is reduced to give the one-dimensional spectrum with high degeneracy, the problem becomes one-dimensional,\textsuperscript{8} whereas the CSM provides trivially the one-dimensional spectrum with quadratic dispersion.\textsuperscript{5–7} Second, since the Wu's work\textsuperscript{2} was motivated in the line of the Haldane's state counting method,\textsuperscript{1} the QSM of Wu was formulated in terms of the state representation. Therefore, the QSM formulation in the momentum representation has been missing except the case of the CSM in one dimension.\textsuperscript{5–7}

However, to study thermal properties of the system such as the equation of state, virial coefficients and the specific heat of an ideal gas with FES in higher dimensions, the QSM formulation in the momentum representation is inevitably necessary since it is the most familiar and powerful method in QSM\textsuperscript{28,29} and the most natural extension of that for Fermi and Bose statistics.

The author\textsuperscript{30,31} has recently presented such a formulation in the momentum representation for an ideal gas with $0 \leq g \leq 1$ — a degenerate $g$-on gas — in any dimension. By using the formulation the equation of state and virial coefficients of all order for the system were successfully obtained generalizing the known results of Sutherland\textsuperscript{6,7} in one dimension and of Wu\textsuperscript{2} in two dimensions. In this respect the theory presented works very well and is a very natural generalization of the QSM in the momentum representation for Fermi and Bose gases.

In this paper we would like to calculate the specific heat of a degenerate $g$-on gas in $D$ dimensions by using the QSM formulation in the $D$-dimensional momentum representation. It is known that for the degenerate Fermi and Bose gases the specific heat of the system is calculated by using the low temperature and high density expansions in terms of the language of Riemann zeta functions.\textsuperscript{28,29} This is known as the so-called Sommerfeld theory, and the expansions as Sommerfeld expansions.\textsuperscript{28,29,32} As we will see later, when we calculate the specific heat of a degenerate ideal $g$-on gas, we find that a class of integrals such as the total energy, the equation of state, the pressure and others for a degenerate ideal $g$-on gas are represented by a generalization of the low temperature and high density expansions in terms of the language of a kind of generalized Riemann zeta functions, which are thought of as being different from the standard definition of the generalized Riemann zeta function.\textsuperscript{33} Therefore, the expansions can be regarded as a generalized Sommerfeld expansion for the $g$-on gas. This is the main objective of the present paper. We also present the high temperature and low density expansions known as the virial expansions.\textsuperscript{28,29}

The organization of the paper is the following: In Sec. 2, we present the QSM formulation for an ideal gas with FES in $D$ dimensions. Here we discuss duality in the theory and some basic theorems. In Sec. 3, we review the degenerate fermion and boson gases for later purposes. In Sec. 4, we discuss the degenerate $g$-on gas. Here we derive the high temperature and low density expansions and the low temperature and high density expansions in terms of the generalized Riemann zeta
function. And using the latter expansions we calculate the specific heat of the \( g \)-on.

gas. In Sec. 5, we generalize the low temperature and high density expansions to the system with a more realistic energy spectrum that is caused by a periodic lattice structure. In Sec. 6, a conclusion is made.

2. QSM of an Ideal Gas with FES in any Dimension

2.1. Haldane’s definition of FES

Haldane first presented a generalized version of Pauli principle using state counting methods. He defined the statistical interactions \( g_{ab} \) through the differential relation:

\[
\Delta D_a = - \sum_b g_{ab} \Delta N_b ,
\]

where \( D_a \) is the dimension of the Hilbert space \( H_a \) of states of a single particle of species \( a \), confined to a finite region of matter and \( N_a \) the number of particles of species \( a \). \( D_a \) can change as particles are added, while keeping the boundary conditions and size of the condensed-matter region constant. By integration it is given as \( D_a = G_a - \sum_b g_{ab}(N_b - \delta_{ab}) \), where \( G_a \) is a constant being interpreted as the number of available single particle states of species \( a \) when no particle is presented in the system.

2.2. QSM for an ideal gas with FES in the state representation: Wu’s method

Wu used the above dimension of the Hilbert space in order to consider the grand partition function \( Q \) of an ideal gas with FES, defined as

\[
Q = \sum_{N=0}^{\infty} \sum_{\{N_a\},N=\sum_a N_a} \left( \prod_a z_a^{N_a} \right) Q_N ,
\]

where \( z_a \) is the fugacity defined by \( z_a = \exp[\beta \mu_a] \) with \( \beta = 1/kT \) (\( k \) is the Boltzmann constant and \( T \) the temperature) and \( \mu_a \) the chemical potential of particles of species \( a \), and \( N \) the total number of particles defined by \( N = \sum_a N_a \). Now, the canonical partition function \( Q_N \) is defined by

\[
Q_N = \sum_{\{N_a\}} W(\{N_a\}) e^{-\beta E(\{N_a\})} ,
\]

\[
W(\{N_a\}) = \prod_a \frac{[D_a + N_a - 1]!}{N_a ! [D_a - 1]!} \equiv e^{S/k} ,
\]

\[
E(\{N_a\}) = \sum_a \varepsilon_a N_a .
\]
Substituting Eqs. (3)–(5) into Eq. (2), Wu first considered the most probable distribution of $Q$, taking the extremum condition: $(\delta / \delta N_a)\{\ln W(\{N_a\}) + \sum_a \beta (\varepsilon_a - \mu_a) N_a\} = 0$, which yields the famous Wu’s distribution function $w_a$:

$$ w_a = \frac{1}{n_a} - \sum_b g'_{a,b} \frac{n_b}{n_a}, \quad (6) $$

$$ (1 + w_a) \prod_b \left( \frac{w_b}{1 + w_b} \right)^{g_{a,b}} = e^{\beta (\varepsilon_a - \mu_a)}, \quad (7) $$

where $n = N_a/G_a$ and $g'_{a,b} \equiv g_{a,b} G_b / G_a$. Especially for an identical FES-particle system taking $g_{a,b} = g \delta_{a,b}$ and $\mu_a = \mu$, Eqs. (6) and (7) turn out to be

$$ n_a = \frac{1}{w_a + g}, \quad w_a^g (1 + w_a)^{1-g} = e^{\beta (\varepsilon_a - \mu)}. \quad (8) $$

Substituting Eqs. (6) and (7) into the thermodynamic potential $\Omega$ and the total number $N$, respectively, we obtain

$$ \Omega = -PV = -kT \sum_a G_a \ln \left( \frac{1 + w_a}{w_a} \right), \quad (9) $$

$$ N = \sum_a G_a n_a, \quad (10) $$

where $n_a$ is defined by Eq. (6) and which is valid for all cases with different species.

2.3. QSM for an ideal gas with FES in the momentum representation

Let us convert the summation over states in Eq. (3) into that over momentum. To do so, we first convert the summation over states in $Q_N$ into that over momentum as

$$ Q_N = \sum_{\{n_p\}, N-\sum_p n_p} g(\{n_p\}) e^{-\beta E(\{n_p\})}, \quad (11) $$

where $g(\{n_p\})$ is the number of states corresponding to $\{n_p\}$.\(^{28}\) Substitution of Eq. (11) into Eq. (2) yields

$$ Q = \sum_{N=0}^{\infty} \sum_{\{n_p\}, N-\sum_p n_p} g(\{n_p\}) z^N e^{-\beta E(\{n_p\})}. \quad (12) $$
When we consider the \( Q \) for Fermi and Bose gases, it is easy to perform the conversion since \( g(\{n_p\}) = 1 \) and we can exchange the summation of \( N \) and that of \( n_p \) such that

\[
Q = \prod_p \sum_{\{n_p\}} (ze^{-\beta \varepsilon_p})^{n_p}.
\]  (13)

In these cases, the summation over \( n_p \) is carried out for \( n_p = 0, 1 \) for fermion and for \( n_p = 0, 1, \ldots, \infty \) for boson. Therefore, the grand partition functions can be factorized as

\[
Q = \prod_p Q_p,
\]  (14)

where \( Q_p = 1 + ze^{-\beta \varepsilon_p} = (1 - ze^{-\beta \varepsilon_p})^{-1} \) for fermion (boson). However, for an ideal gas with FES, even if we impose the same relation \( g(\{n_p\}) = 1 \), the procedure is not that simple since we must have another constraint coming from the constraint for \( D_o \). Thus, the problem becomes much harder. But if possible, what is \( Q_p \) for an ideal gas with FES? To answer this, instead of using the above standard procedure we follow the Wu's argument as an inverse method.

To convert the summation over states in Eqs. (9) and (10) into that over momentum, let us first suppose the factorized form [Eq. (14)] of the grand partition function at the most probable distribution. From this, we can naturally define the average occupation number \( \langle n_p \rangle \) at momentum \( p \) in the momentum representation as

\[
\langle n_p \rangle = -\frac{\partial}{\partial (\beta \varepsilon_p)} \ln Q = -\frac{\partial}{\partial (\beta \varepsilon_p)} \ln Q_p.
\]  (15)

Then we inversely ask whether we can obtain the expression of \( Q_p \) in terms of \( \langle n_p \rangle \). This can be carried out as follows: Let us define as

\[
\langle n_p \rangle \equiv \frac{1}{W_p + g}
\]  (16)

at the most probable distribution and substituting this into Eq. (15), we can integrate it with respect to \( \beta \varepsilon_p \). For an ideal gas, using the relation:

\[
W_p^g(1 + W_p)^{1-g} = e^{\beta (\varepsilon_p - \mu)},
\]  (17)

we end up with \( Q_p = (1 + W_p)/W_p \), where we have used simple relations: \( \beta (\varepsilon_p - \mu) = g \ln W_p + (1 - g) \ln (1 + W_p) \), and \( d(\beta \varepsilon_p) = (W_p + g)/(W_p(1 + W_p))dW_p \). Hence, we obtain the thermodynamic potential \( \Omega \) and the total number \( N \) in the momentum representation:

\[
\Omega = -PV = -kT \sum_p \ln \left( \frac{1 + W_p}{W_p} \right),
\]  (18)
We note that the above Eqs. (18)-(20) are generalized to those for the more general cases with mutual statistics \( g_{p,q} \) as well, using the following relations:

\[
\sum_q \langle n_q \rangle (W_p g_{p,q} + g_{p,q}) = 1,
\]

\[
(1 + W_p) \prod_q \left( \frac{W_q}{1 + W_q} \right)^{g_{p,q}} = e^{\beta (\varepsilon_p - \mu)}.
\]

2.4. **QSM for an ideal gas with FES in the energy representation**

Let us convert the momentum representation into the energy representation. Converting the summation into integration as \( \sum_p = [V/(2\pi \hbar)^D] \int d^D p = [V/(2\pi \hbar)^D] S_D \int p^{D-1} dp \), where \( S_D = 2\pi^{D/2}/\Gamma(D/2) \) and \( \Gamma(s) \) is the Gamma function. Assuming free FES particles with \( \varepsilon_p = p^2/2m \), the density of states (DOS) is given from \((1/V) \sum_p = \int d\varepsilon N_D(\varepsilon)\) as

\[
N_D(\varepsilon) = \left( \frac{m}{2\pi \hbar} \right)^{D/2} \frac{1}{\Gamma(D/2)} \varepsilon^{(D-2)/2}.
\]

Now, Eqs. (18)–(20) are represented with respect to the energy \( \varepsilon \) as

\[
\frac{P}{kT} = \int_0^\infty d\varepsilon N_D(\varepsilon) \ln \left[ \frac{1 + W(\beta(\varepsilon - \mu))}{W(\beta(\varepsilon - \mu))} \right],
\]

\[
\frac{N}{V} = \frac{1}{v} = \int_0^\infty d\varepsilon N_D(\varepsilon) \frac{1}{W(\beta(\varepsilon - \mu)) + g},
\]

\[
\frac{E}{V} = \int_0^\infty d\varepsilon N_D(\varepsilon) \varepsilon \frac{1}{W(\beta(\varepsilon - \mu)) + g},
\]

where

\[
W(\beta(\varepsilon - \mu))\varepsilon[1 + W(\beta(\varepsilon - \mu))]^{1-g} = e^{\beta(\varepsilon - \mu)}.
\]
2.5. Duality in the theory

We note that there are some symmetries in the theory. Defining \( z = \exp(\beta \mu) \), let us denote \( P \) in Eq. (23), \( N \) in Eq. (24), and \( W(\beta(\varepsilon - \mu)) \) in Eq. (25) by \( P(T, z, g) \), \( N(T, z, g) \), and \( W(T, z, g) \), respectively. First, there is a duality between statistics \( g \) and \( 1/g \), which is associated with interchange between particle and hole\(^{14,15}\) and yields

\[
W(T, z, 1/g) = \frac{1}{W(-T/g, z, g)}. \tag{27}
\]

By substitution into Eq. (24) and with some algebra we derive

\[
\frac{N(T, z, 1/g)}{V} = g \left( 1 - g \frac{N(-T/g, z, g)}{V} \right), \tag{28}
\]

which was first derived by Nayak and Wilczek [see Eq. (18) in Ref. 15].

Second, there is another duality between statistics \( g \) and \( 1 - g \), which was first discussed by Sutherland\(^6\) and is associated with supersymmetry in a superpair of \( g \)-on and \( (1 - g) \)-on. This yields

\[
W(T, z, 1 - g) = -W(T, -z, g) - 1, \tag{29}
\]

and hence by substitution we derive

\[
P(T, z, 1 - g) = P(T, -z, g). \tag{30}
\]

2.6. Some basic theorems

Let us consider some basic theorems.

**Theorem 1.** Denote the pressure by \( P/kT = (1/V) \sum_p \ln[(1 + W_p)/W_p] \equiv (1/\lambda^D)F(z) \). Then

\[
\frac{N}{V} = \frac{1}{V} \sum_p \langle n_p \rangle = \frac{1}{\lambda^D} z \frac{\partial}{\partial z} F(z), \tag{31}
\]

where \( \lambda \) is the thermal length defined by \( \lambda \equiv \sqrt{2\pi\hbar^2/mkT} \).

This theorem is well-known and a classic result in QSM.\(^{28}\) However, to show that it indeed holds valid for ideal FES gases is not so trivial since there is the nontrivial Wu's distribution. Differentiating \( F(z) \) with respect to \( z \), we find that

\[
z \frac{\partial}{\partial z} F(z) = \frac{1}{V} \sum_p \frac{-1}{(1 + W_p)W_p} z \frac{\partial W_p}{\partial z}. \tag{32}
\]
Differentiating Eq. (17) with respect to $z$, we find

$$\frac{\partial W_p}{\partial z} = \frac{-W_p(1 + W_p)}{W_p + g}. \quad (33)$$

Hence, by substitution we prove the theorem.

**Theorem 2.** The second order fluctuation $\langle (\Delta n_p)^2 \rangle \equiv \langle n_p^2 \rangle - \langle n_p \rangle^2 = -[\partial / \partial (\beta \varepsilon_p)] \langle \xi_{np} \rangle$ is given by

$$\langle (\Delta n_p)^2 \rangle = \langle n_p \rangle (1 - g \langle n_p \rangle) (1 + (1 - g) \langle n_p \rangle) \quad (34)$$

for an ideal $g$-on gas.

This theorem interpolates between classic results of fermion and boson cases.\textsuperscript{28} And this type of relations were first noted by Nayak and Wilczek,\textsuperscript{15} and derived by Rajagopal.\textsuperscript{17} The proof of this theorem is straightforward using Eq. (17) as follows:

Differentiate $\langle n_p \rangle$ with respect to be $\beta \varepsilon_p$. We find that

$$\frac{\partial}{\partial (\beta \varepsilon_p)} \langle n_p \rangle = \frac{\partial}{\partial (\beta \varepsilon_p)} \frac{1}{W_p + g} = -\frac{1}{(W_p + g)^2} \frac{\partial W_p}{\partial \beta \varepsilon_p}. \quad (35)$$

Substituting Eq. (33) together with the sign difference of $\partial W_p / \partial (\beta \varepsilon_p) = -z(\partial W_p / \partial z)$, we get

$$- \frac{\partial}{\partial (\beta \varepsilon_p)} \langle n_p \rangle = \frac{W_p(1 + W_p)}{(W_p + g)^3} = \frac{1}{W_p + g} \frac{W_p}{W_p + g} \frac{1 + W_p}{W_p + g}$$

$$= \frac{1}{W_p + g} \left(1 - \frac{g}{W_p + g}\right) \left(1 + \frac{1 - g}{W_p + g}\right)$$

$$= \langle n_p \rangle (1 - g \langle n_p \rangle) (1 + (1 - g) \langle n_p \rangle). \quad (36)$$

### 3. Degenerate Fermi and Bose Gases and the Riemann Zeta Function

In this section we review the theory of degenerate fermion and boson gases in $D$ dimensions. We present both the high temperature and low density expansions and the low temperature and high density expansions for the later purposes. The former is known as the virial expansions,\textsuperscript{28,29} while the latter is known as the Sommerfeld expansions.\textsuperscript{32} Therefore, there is nothing new in this section but it is necessary for comparison between the theory of fermion and boson gases and that of $g$-on gases.

When $g = 1$ ($= 0$) the above equations turn out to the famous expressions for a degenerate Fermi (Bose) gas. From Eq. (26), we find that if $g = 1$ then $W(\beta (\varepsilon - \mu)) = e^{\beta (\varepsilon - \mu)}$ and if $g = 0$ then $W(\beta (\varepsilon - \mu)) = e^{\beta (\varepsilon - \mu)} - 1$. Thus, the
Eqs. (23)–(26) become

\[
P \frac{kT}{kT} = \pm \int_0^\infty d\varepsilon N_D(\varepsilon) \ln \left( 1 \pm e^{-\beta(\varepsilon - \mu)} \right)
= \pm \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2)} \int_0^\infty d\varepsilon \ v^{(D-2)/2} \ln \left( 1 \pm e^{-\beta(\varepsilon - \mu)} \right)
\]

(37)

\[
N \frac{V}{V} = \int_0^\infty d\varepsilon N_D(\varepsilon) e^{\beta(\varepsilon - \mu)} = 1
= \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2)} \int_0^\infty d\varepsilon \ v^{(D-2)/2} \frac{1}{e^{\beta(\varepsilon - \mu)} + 1}
\]

(38)

\[
E \frac{V}{V} = \int_0^\infty d\varepsilon N_D(\varepsilon) e^{\beta(\varepsilon - \mu)} = 1
= \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2)} \int_0^\infty d\varepsilon \ v^{(D-2)/2} \frac{1}{e^{\beta(\varepsilon - \mu)} + 1}
\]

(39)

where \( \mu \) must be negative for bosons. Integration by parts of Eq. (37) provides

\[
P = \frac{2}{D} \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2)} \int_0^\infty d\varepsilon \ v^{(D-2)/2} \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} = \frac{2}{D} \frac{E}{V}
\]

(40)

In this way, for the degenerate Fermi and Bose gases one must treat integrals of the following type:

\[
I = \int_0^\infty d\varepsilon \ \frac{f(\varepsilon)}{e^{\beta(\varepsilon - \mu)} + 1}
\]

(41)

3.1. High temperature and low density expansions

Let us consider the high temperature and low density limit of \( \lambda^D/v \ll 1 \). In this limit we expand the exponential part in the integrand as

\[
\frac{1}{e^{\beta(\varepsilon - \mu)} + 1} = \sum_{l=1}^\infty (-1)^l e^{-l\beta(\varepsilon - \mu)}
\]

(42)

and substituting into Eq. (41) we obtain

\[
I = \sum_{l=1}^\infty (-1)^l \frac{1}{z} \int_0^\infty d\varepsilon f(\varepsilon) e^{-l\beta \varepsilon}
\]

(43)

where \( z = \exp(\beta \mu) \). In Eq. (43), the integration is nothing more than the Laplace transform \( \mathcal{L}[f(l\beta)] \) of the function \( f(\varepsilon) \), where \( \mathcal{L}[f(l\beta)] = \int_0^\infty d\varepsilon f(\varepsilon) e^{-l\beta \varepsilon} \).
Hence, we obtain

$$I = \sum_{l=1}^{\infty} (\pm 1)^{l+1} L[f(l\beta)] z^l.$$  \hspace{1cm} (44)

If we apply the above formula for the pressure, the density, and the energy, respectively, using the DOS of Eq. (22), then we find

$$\frac{P}{kT} = \frac{2}{D} \frac{1}{kT} \frac{E}{V} = \frac{1}{\lambda^D} \sum_{l=1}^{\infty} b_l z^l,$$  \hspace{1cm} (45)

$$\frac{N}{V} = \frac{1}{\lambda^D} \sum_{l=1}^{\infty} lb_l z^l,$$  \hspace{1cm} (46)

where \( b_l = (\pm 1)^{l+1}/l^{D/2+1} \) and \( + \) (−) sign is taken for boson (fermion). If we solve Eq. (46) for \( z \) in terms of \( \lambda^D/v \), then we can express \( P \) in terms of \( \lambda^D/v \) such as

$$\frac{PV}{NkT} = \frac{\sum_{l=1}^{\infty} b_l z^l}{\sum_{l=1}^{\infty} lb_l z^l} = \sum_{l=1}^{\infty} B_l \left( \frac{\lambda^D}{v} \right)^l,$$  \hspace{1cm} (47)

which is called the virial expansion and \( B_l \) the virial coefficients.\(^{28,29}\) Using the standard argument,\(^28\) the virial coefficients \( B_l \) are represented in terms of \( b_l \) such as

\[
\begin{align*}
B_1 &= b_1 = 1, \\
B_2 &= -b_2, \\
B_3 &= 4b_2^2 - 2b_3, \\
B_4 &= -20b_2^3 + 18b_2b_3 - 3b_4.
\end{align*}
\]  \hspace{1cm} (48)

### 3.2. Low temperature and high density expansions

Let us consider another limit — the low temperature and high density limit — of \( \lambda^D/v \gg 1 \). Substituting \( x = \beta (\varepsilon - \mu) \) into Eq. (41) transforms the integral to

\[
\begin{align*}
I &= kT \int_{-\beta \mu}^{\infty} dx \frac{f(\mu + x/\beta)}{e^x + 1} \\
&= kT \int_{0}^{\beta \mu} dx \frac{f(\mu - x/\beta)}{e^{-x} + 1} + kT \int_{0}^{\infty} dx \frac{f(\mu + x/\beta)}{e^x + 1} \\
&= \int_{0}^{\mu} dx f(x) - kT \int_{0}^{\beta \mu} dx \frac{f(\mu - x/\beta)}{e^{-x} + 1} + kT \int_{0}^{\infty} dx \frac{f(\mu + x/\beta)}{e^x + 1},
\end{align*}
\]  \hspace{1cm} (49)

where we have used the relation: \( 1/(e^{-x} + 1) = 1 - 1/(e^x + 1) \).
Let us first consider fermions. In this limit for fermions, the evaluation of the
integral at low temperature is well known as the Sommerfeld theory for a metal. In
the second integral of Eq. (49), we replace the upper limit by infinity, since
\( \beta \mu \gg 1 \) for fermions and the integral is rapidly convergent. This gives

\[
I_F = \int_0^\mu dx f(x) + kT \int_0^\infty dx \frac{f(\mu + x/\beta) - f(\mu - x/\beta)}{e^x + 1}.
\]  

(50)

By Taylor expansions of \( f(\mu + x/\beta) \) and \( f(\mu - x/\beta) \) we obtain

\[
f\left( \mu + \frac{x}{\beta} \right) - f\left( \mu - \frac{x}{\beta} \right) = 2 \left[ \frac{x}{\beta} f'(\mu) + \frac{1}{3!} \left( \frac{x}{\beta} \right)^3 f'''(\mu) + \cdots \right].
\]  

(51)

Hence we obtain the following formula:

\[
I_F = \int_0^\mu dx f(x) + 2(kT)^2 \times \left[ f'(\mu) \int_0^\infty dx \frac{x}{e^x + 1} + \frac{1}{3!} (kT)^2 f'''(\mu) \int_0^\infty dx \frac{x^3}{e^x + 1} + \cdots \right].
\]  

(52)

As is known, the integrals in Eq. (52) is represented in terms of the Gamma
function \( \Gamma(s) \) and the Riemann zeta function \( \zeta(s) \) satisfying the following relations:

\[
\int_0^\infty dx \frac{x^{s-1}}{e^x + 1} = (1 - 2^{1-s})\Gamma(s)\zeta(s),
\]  

(53)

\[
\int_0^\infty dx \frac{x^{s-1}}{e^x - 1} = \Gamma(s)\zeta(s),
\]  

(54)

where the Riemann zeta function\(^{32}\) is defined as

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]  

(55)

Thus, we obtain

\[
I_F = \int_0^\mu dx f(x) + 2 \sum_{s=1}^\infty (kT)^{2s} (1 - 2^{1-2s})\zeta(2s)f^{(2s-1)}(\mu).
\]  

(56)

This is known as the Sommerfeld expansion\(^{28,29,32}\) We place here some known
values of \( \zeta(2s) \): \( \zeta(2) = \pi^2/6 \), \( \zeta(4) = \pi^4/90 \), \( \zeta(6) = \pi^6/945 \).

For bosons, on the other hand, Eq. (41) yields

\[
I_B = \frac{d_0}{e^{-\beta\mu} - 1} + kT \int_{-\beta\mu}^{\infty} dx \frac{f(\mu + x/\beta)}{e^x - 1} = I_{B,0} + I_{B,1},
\]  

(57)
where the first term $I_{B,0}$ is a contribution from $\epsilon = 0$ with $d_0$ being the degeneracy at this state and the second term comes from excited states, and we have assumed that $f(x)$ satisfies $f(0) = 0$. Since $\mu$ is very close to 0 (i.e. $\mu = 0$), the second term becomes

$$I_{B,1} = kT \int_0^{\infty} dx \frac{f(x/\beta)}{e^{\epsilon} - 1},$$

while the first term can contribute only when $f(0) = 0$, otherwise it has no contribution but is included in the second integral. For example, if we assume that $f(x) \propto x^{s-1}$, then using Eq. (54) we obtain

$$I_{B,2} = (kT)^s \int_0^{\infty} dx \frac{x^{s-1}}{e^{\epsilon} - 1} = (kT)^s \Gamma(s)\zeta(s).$$

Thus, we finally obtain

$$I_B = \frac{d_0}{e^{-\beta\mu} - 1} \theta(s - 1) + (kT)^s \Gamma(s)\zeta(s),$$

where $\theta(s - 1)$ is a step function such that $\theta(s - 1) = 1 (= 0)$ for $s > 1$ (otherwise).

3.3. Specific heat of a degenerate Fermi gas

For example, using the Eq. (56) for Eq. (39), we are able to calculate the specific heat of a degenerate Fermi gas as follows: If we assign $f(\epsilon)$ as $f(\epsilon) = (m/2\pi\hbar^2)^{D/2} [1/\Gamma(D/2)]e^{\epsilon/2}$, then the energy $E$ is given by

$$E = \frac{D}{2} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \left[ \frac{1}{\Gamma(D/2 + 2)} \mu^{D/2 + 1} + \frac{(kT)^2}{\Gamma(D/2 - 2)} \mu^{D/2 - 3} + \ldots \right]$$

$$= \frac{D}{2} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \left( \frac{1}{\Gamma(D/2 + 2)} \mu^{D/2 + 1} \right)$$

$$\times \left[ 1 + 2 \sum_{s=1}^{\infty} (1 - 2^{1-2s}) \zeta(2s) \frac{\Gamma(D/2 + 2)}{\Gamma(D/2 + 2 - 2s)} \left( \frac{kT}{\mu} \right)^{2s} \right]$$

$$= \frac{D}{2} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \left( \frac{1}{\Gamma(D/2 + 2)} \mu^{D/2 + 1} \right)$$

$$\times \left[ 1 + \zeta(2) \left( \frac{D}{2} + 1 \right) \frac{kT}{\mu} \right] + \ldots. \quad (61)$$
Similarly, the density is represented as

\[
\frac{N}{V} = \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \left( \frac{1}{\Gamma(D/2 + 1)} \mu^{D/2} \right)
\times \left[ 1 + 2 \sum_{s=1}^{\infty} (1 - 2^{1-2s}) \zeta(2s) \frac{\Gamma(D/2 + 1)}{\Gamma(D/2 + 2 - 2s)} \left( \frac{kT}{\mu} \right)^{2s} \right]
\]

\[
= \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \left( \frac{1}{\Gamma(D/2 + 1)} \mu^{D/2} \right) \left[ 1 + \zeta(2) \frac{D}{2} \left( \frac{D}{2} - 1 \right) \left( \frac{kT}{\mu} \right)^2 + \cdots \right]. \quad (62)
\]

At \( T = 0 \) the chemical potential becomes the Fermi energy as \( \mu = \varepsilon_F \) and since the Fermi distribution function becomes unity such as \( \langle n_p \rangle = 1 \). Therefore, we have

\[
\frac{N}{V} = \frac{1}{V} \sum_p \langle n_p \rangle = \frac{1}{(2\pi \hbar)^D} \int_{0}^{\varepsilon_F} d^D p
\]

\[
= \int_{0}^{\varepsilon_F} dx N_D(x) = \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2 + 1)} \varepsilon_F^{D/2}, \quad (63)
\]

\[
\frac{E}{V} = \frac{1}{V} \sum_p \varepsilon_p \langle n_p \rangle = \frac{1}{(2\pi \hbar)^D} \int_{0}^{\varepsilon_F} d^D p \frac{p^2}{2m}
\]

\[
= \int_{0}^{\varepsilon_F} dx x N_D(x) = \frac{D}{2} \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2 + 2)} \varepsilon_F^{D/2+1}
\]

\[
= \frac{D}{D+2} \frac{N}{V} \varepsilon_F. \quad (64)
\]

Using Eq. (63) for Eq. (62), we obtain

\[
\varepsilon_F^{D/2} = \mu^{D/2} \left[ 1 + 2 \sum_{s=1}^{\infty} (1 - 2^{1-2s}) \zeta(2s) \frac{\Gamma(D/2 + 1)}{\Gamma(D/2 + 2 - 2s)} \left( \frac{kT}{\mu} \right)^{2s} \right]
\]

\[
= \mu^{D/2} \left[ 1 + \zeta(2) \frac{D}{2} \left( \frac{D}{2} - 1 \right) \left( \frac{kT}{\mu} \right)^2 + \cdots \right]. \quad (65)
\]

Solving for \( \mu \) in terms of \( \varepsilon_F \), we get

\[
\mu = \varepsilon_F \left[ 1 - \zeta(2) \left( \frac{D}{2} - 1 \right) \left( \frac{kT}{\varepsilon_F} \right)^2 + \cdots \right]. \quad (66)
\]
Substituting this into Eq. (61) and expanding with respect to $kT/\varepsilon_F$, we get

$$E = \frac{D}{D+2}N\varepsilon_F \left[ 1 + \zeta(2) \left( \frac{D}{2} + 1 \right) \left( \frac{kT}{\varepsilon_F} \right)^2 + \cdots \right]. \quad (67)$$

Hence, by differentiation with respect to $T$ we obtain the specific heat $C_v = (\partial E/\partial T)_V$ with linear $T$ dependence as

$$C_v = \left( \frac{\partial E}{\partial T} \right)_V = \frac{D}{D+2} N\varepsilon_F \zeta(2) \left( \frac{D}{2} + 1 \right) 2 \left( \frac{k}{\varepsilon_F} \right)^2 T + O(T^3)$$

$$= DNk \zeta(2) \left( \frac{kT}{\varepsilon_F} \right) + O(T^3) = \frac{\pi^2}{6} DNk \left( \frac{kT}{\varepsilon_F} \right) + O(T^3). \quad (68)$$

4. Degenerate g-on Gas and the Generalized Riemann Zeta Function

Let us now consider the integrals for a degenerate g-on gas. As in the above discussion for Fermi and Bose gases, the integration by parts converted the expression of Eq. (37) into the form of Eqs. (38) and (39) with the occupation number. Similarly, for the g-on gas, the integration of pressure [Eq. (23)] is converted into the same form of Eqs. (24) and (25) although there is the Wu’s distribution function $W(\varepsilon)$ in the integrand as follows: Applying integration by parts, the pressure then becomes

$$\frac{P}{kT} = \int_0^\infty d\varepsilon N_D(\varepsilon) \ln \left[ \frac{1 + W(\beta(\varepsilon - \mu))}{W(\beta(\varepsilon - \mu))} \right]$$

$$= \int_0^\infty d\varepsilon (M_D(\varepsilon)) \ln \left[ \frac{1 + W(\beta(\varepsilon - \mu))}{W(\beta(\varepsilon - \mu))} \right]$$

$$= \left[ M_D(\varepsilon) \ln \left[ \frac{1 + W(\beta(\varepsilon - \mu))}{W(\beta(\varepsilon - \mu))} \right] \right]_0^\infty - \int_0^\infty d\varepsilon M_D(\varepsilon)$$

$$\times \left[ \frac{-1}{W(\beta(\varepsilon - \mu))(1 + W(\beta(\varepsilon - \mu))))} \right] \frac{\partial W(\beta(\varepsilon - \mu))}{\partial \varepsilon},$$

$$= \beta \int_0^\infty d\varepsilon M_D(\varepsilon) \frac{1}{W(\beta(\varepsilon - \mu)) + g}; \quad (69)$$

where we have defined as $M_D(\varepsilon) = \int d\varepsilon N_D(\varepsilon)$ and used the relation:

$$\frac{\partial W(\beta(\varepsilon - \mu))}{\partial \varepsilon} = \beta \frac{W(1 + W)}{W + g}. \quad (70)$$
In this way we need to consider the following type of integrals for a degenerate $g$-on gas:

$$I = \int_0^\infty d\varepsilon \frac{f(\varepsilon)}{W(\beta(\varepsilon - \mu)) + g}.$$  \hspace{1cm} (71)

Using Eq. (69) and the DOS of Eq. (22), we also find the relation: $P = (2/D)E/V$ for a $D$-dimensional ideal degenerate $g$-on gas.

4.1. **High temperature and low density expansions**

Let us consider the high temperature and low density limit of $\lambda^D/v \ll 1$ for a degenerate $g$-on gas. In this limit, the above integral can be calculated by means of the fugacity expansions as follows: Denote as

$$\xi(\beta(\varepsilon - \mu)) = 1 + \frac{1}{W(\beta(\varepsilon - \mu))}. \hspace{1cm} (72)$$

By this replacement, Eq. (26) is converted into

$$\xi(\beta(\varepsilon - \mu))^2 - \xi(\beta(\varepsilon - \mu))^{2-1} = e^{\beta(\mu - \varepsilon)}, \hspace{1cm} (73)$$

which was first found by Sutherland\(^6,7\) for the CSM and used recently by Dasnieres de Veigy and Ouvry\(^8\) for the fractional quantum Hall system. Then we find

$$z \frac{\partial}{\partial z} \ln \xi(\beta(\varepsilon - \mu)) = \frac{1}{W(\beta(\varepsilon - \mu)) + g}. \hspace{1cm} (74)$$

So, instead of Eq. (71), we would like to consider an integral of the following type:

$$J = \int_0^\infty d\varepsilon f(\varepsilon) \ln \xi(\beta(\varepsilon - \mu)) \hspace{1cm} (75)$$

such that $I = z(\partial J/\partial z)$.

Following the method of Sutherland,\(^6\) $\ln \xi(x)$ can be expanded with respect to $x$ as

$$\ln \xi(x) = \sum_{l=1}^\infty c_l e^{-lx}, \hspace{1cm} (76)$$

where the coefficients $c_l$ are given by

$$c_l = (-1)^{l+1} \frac{[lg]!}{l!} \frac{l!}{[(lg - 1)]!} = (-1)^{l+1} \frac{lg^{l+1}}{l!} (lg - 1) \cdots (lg - (l - 2))(lg - (l - 1)). \hspace{1cm} (77)$$
For example, we have the first four coefficients as
\[
\begin{align*}
    c_1 &= 1, \\
    c_2 &= \frac{2g - 1}{2!}, \\
    c_3 &= \frac{(3g - 2)(3g - 1)}{3!}, \\
    c_4 &= \frac{(4g - 3)(4g - 2)(4g - 1)}{4!}.
\end{align*}
\]

(78)

Here we note that the coefficients become \(c_1 = 1/l\) for \(g = 0\) and \(c_2 = (-1)^{l+1}/l\) for \(g = 1\). The above expansion of Eq. (76) is nothing more than the *cluster expansion* in statistical mechanics\(^{28,29}\) (but if we would like to put the restricted meaning on the expansion for the FES system, then we may call the above expansion the *Sutherland expansion*.) We note here that another derivation for the above expansion using Lagrange's theorem\(^{33}\) has appeared recently.\(^{34}\)

Substituting Eq. (76) into Eq. (75) yields
\[
J = \sum_{l=1}^{\infty} c_l z^l \int_0^\infty d\varepsilon f(\varepsilon)e^{-l\beta \varepsilon} = \sum_{l=1}^{\infty} c_l z^l \mathcal{L}[f(l\beta)],
\]

(79)

where the last integral \(\mathcal{L}[f(\varepsilon)] = \int_0^\infty d\varepsilon f(\varepsilon)e^{-\varepsilon}e^{-l\beta \varepsilon}\) is the Laplace transform of the function \(f(\varepsilon)\) as before. Hence, the integral \(I\) of Eq. (71) is given by
\[
I = \sum_{l=1}^{\infty} l c_l \mathcal{L}[f(l\beta)]z^l.
\]

(80)

To find the high temperature expansion of \(I\) (or \(J\)) we have to find that of \(z\). From considering Eq. (24), we find
\[
\frac{N}{V} = \int_0^\infty d\varepsilon N_D(\varepsilon) \frac{1}{W(\beta(\varepsilon - \mu)) + g} = \sum_{l=1}^{\infty} l c_l \mathcal{L}[N_D(l\beta)]z^l.
\]

(81)

If we use the DOS of Eq. (22) for the above, then we find
\[
\mathcal{L}[N_D(l\beta)] = \left(\frac{m}{2\pi \hbar^2}\right)^{D/2} \frac{1}{\Gamma(D/2)} \int_0^\infty d\varepsilon \varepsilon^{(D-2)/2} e^{-l\beta \varepsilon} = \frac{1}{\lambda^D} \frac{1}{l^{D/2}}.
\]

(82)

Substituting Eq. (82) into Eq. (81), we obtain
\[
\gamma = \frac{\lambda^D}{v} = \sum_{l=1}^{\infty} l b_l z^l = b_1 z + 2 b_2 z^2 + 3 b_3 z^3 + \cdots.
\]

(83)
4.2. Low temperature and high density expansions

Let us consider the low temperature and high density limit of $\lambda^D/\nu \gg 1$ for the degenerate $g$-on gas. In this limit, as for the cases of Fermi and Bose gases, the substitution $x = \beta(\varepsilon - \mu)$ with $\beta \mu > 1$ gives

$$ I = kT \int_{-\beta \mu}^{\infty} dx \frac{f(\mu + x/\beta)}{W(x) + g} $$
$$ = kT \int_{0}^{\beta \mu} dx \frac{f(\mu - x/\beta)}{W(-x) + g} + kT \int_{0}^{\infty} dx \frac{f(\mu + x/\beta)}{W(x) + g}. \quad (89) $$

Let us now consider the relationship between $W(-x)$ and $W(x)$. Let us assume $0 < g < 1$ for later purposes, since the $g = 0$ and 1 cases are extracted from the above, respectively, and coincide with the results of the previous section. From Eq. (26), we assume that $W(x)$ and $W(-x)$ satisfy

$$ W(x)^g(1 + W(x))^{1-g} = e^x, \quad (90) $$
$$ W(-x)^g(1 + W(-x))^{1-g} = e^{-x}, \quad (91) $$

respectively. From the particle–hole duality of Eq. (27) and substituting $-T$ into it, we find $W(-T, z, 1/g) = 1/W(T/g, z, g)$. And replacing $g$ by $1/g$, we find $W(-T, z, g) = 1/W(gT, z, 1/g)$. Here $W(gT, z, 1/g)$ and $W(-T, z, g)$ satisfy the relations:

$$ W(gT, z, 1/g)^{1/g}[1 + W(gT, z, 1/g)]^{1-1/g} = e^{\beta(\varepsilon - \mu)/g}, \quad (92) $$
$$ W(-T, z, g)^{g}[1 + W(-T, z, g)]^{1-g} = e^{-\beta(\varepsilon - \mu)/g}, \quad (93) $$

respectively. Thus, if we substitute $x = \beta(\varepsilon - \mu)$ in the above expressions, then they turn out to Eqs. (90) and (91), and hence we find the relation between $W(x)$ and $W(-x)$ as

$$ W(-x) = \frac{1}{W'([x/g])}, \quad (94) $$

where $W'(x/g)$ satisfies

$$ \left[ \frac{W'(x/g)}{g} \right]^{1/g} \left[ 1 + W'\left( \frac{x}{g} \right) \right]^{1-1/g} = e^{x/g}. \quad (95) $$
From Eq. (94), we find
\[
\frac{1}{W(-x)+g} = \frac{W'(x/g)}{g W'(x/g) + 1} = \frac{1}{g} - \frac{1}{g^2 W''(x/g) + 1/g},
\]
which is essentially equivalent to Eq. (28) apart from \( g \) being replaced by \( 1/g \).

Substituting this into Eq. (89), we obtain
\[
I = \frac{1}{g} \int_0^\mu dx f(x) + kT \int_0^\infty dx \left\{ \frac{f(\mu + x/\beta)}{W(x) + g} - \frac{1}{g^2} \frac{f(\mu - x/\beta)}{W'(x/g) + 1/g} \right\}
= \frac{1}{g} \int_0^\mu dx f(x) + kT \int_0^\infty dx \left\{ \frac{f(\mu + x/\beta)}{W(x) + g} - \frac{1}{g} \frac{f(\mu - gx/\beta)}{W'(x) + 1/g} \right\},
\]
where in the last step we have used change of the variable, \( x' = x/g \). If we use the notations: \( n_g(x) = 1/(W_g(x) + g) \) and \( n_{1/g}(x) = 1/(W_{1/g}(x) + 1/g) \), then we can rewrite as
\[
I = \frac{1}{g} \int_0^\mu dx f(x) + kT \int_0^\infty dx \left\{ f(\mu + x/\beta)n_g(x) - \frac{1}{g} f(\mu - gx/\beta)n_{1/g}(x) \right\}.
\]

We now have Taylor expansions of \( f(\mu + x/\beta) \) and \( f(\mu - gx/\beta) \) and the Sutherland expansion of Eq. (76) for \( n_g(x) \) and \( n_{1/g}(x) \):
\[
f(\mu + x/\beta) = f(\mu) + \frac{x}{\beta} f'(\mu) + \frac{1}{2!} \left( \frac{x}{\beta} \right)^2 f''(\mu) + \frac{1}{3!} \left( \frac{x}{\beta} \right)^3 f'''(\mu) + \cdots,
\]
\[
f(\mu - gx/\beta) = f(\mu) - \frac{gx}{\beta} f'(\mu) + \frac{g^2}{2!} \left( \frac{x}{\beta} \right)^2 f''(\mu) - \frac{g^3}{3!} \left( \frac{x}{\beta} \right)^3 f'''(\mu) + \cdots,
\]
\[
n_g(x) = \sum_{l=1}^\infty l c_l(g)e^{-lx},
\]
\[
n_{1/g}(x) = \sum_{l=1}^\infty l c_l(1/g)e^{-lx},
\]
respectively. For example, substituting them into the second integral of Eq. (98) we obtain
Similarly, we obtain for the third integral of Eq. (98)

$$\int_0^\infty dx f(\mu + x/\beta) \eta_0(x) = f(\mu) \sum_{l=1}^{\infty} c_l(g) \int_0^\infty dx e^{-lx}$$

$$+ kT f'(\mu) \sum_{l=1}^{\infty} c_l(g) \int_0^\infty dx xe^{-lx}$$

$$+ \frac{1}{2!} f''(\mu) (kT)^2 \sum_{l=1}^{\infty} c_l(g) \int_0^\infty dx x^2 e^{-lx}$$

$$+ \frac{1}{3!} f'''(\mu) (kT)^3 \sum_{l=1}^{\infty} c_l(g) \int_0^\infty dx x^3 e^{-lx} + \ldots$$

$$= f(\mu) \sum_{l=1}^{\infty} c_l(g) + kT f'(\mu) \sum_{l=1}^{\infty} \frac{c_l(g)}{l}$$

$$+ (kT)^2 f''(\mu) \sum_{l=1}^{\infty} \frac{c_l(g)}{l^2} + \ldots$$  \hspace{1cm} (103)

Substituting these into Eq. (98), we obtain the following expansion at low temperature and high density:

$$I = \frac{1}{g} \int_0^\mu dx f(x) + \sum_{s=1}^{\infty} (kT)^s a_g(s) f^{s-1}(\mu),$$  \hspace{1cm} (105)

where we have defined as

$$a_g(s) = \sum_{l=1}^{\infty} \frac{\chi_l(s)}{l^{s-1}},$$  \hspace{1cm} (106)

$$\chi_l(s) = c_l(g) + (-g)^{s-2} c_l(1/g).$$  \hspace{1cm} (107)
Let us consider some remarkable properties of the coefficients $a_g(s)$. We first find that if $g = 1$, then $\chi_l(s) = 2(-1)^{l+1}/l$ for $s$ even ($= 0$ for $s$ odd), and therefore, Eq. (107) with $g = 1$ coincides with the result of the degenerate ideal Fermi gas case. Thus, Eq. (107) works even for the fermion case as well, although it does not work for the boson case, since we have used the particle-hole duality between $g$ and $1/g$ which is valid only when $g > 0$.

Second, we find that

$$a_g(1) = 0 \text{ for any } g,$$

(108)

which was first conjectured as a consequence of numerical calculations by Nayak and Wilczek\textsuperscript{15} and also investigated by Isakov et al\textsuperscript{24,25} We conclude that this relation is a consequence of neutrality of the system as follows: Consider Eq. (98) and the $T$-linear term is given as

$$I_1 = kT f(\mu) \int_0^\infty dx \left\{ n_g(x) - \frac{1}{g} n_{1/g}(x) \right\} = a_g(1) kT f(\mu).$$

(109)

The first integral is the total number (or charge) of particles in the system such as $N_p = \int_0^\infty dx n_g(x)$, while the second integral the total number (or charge) of holes in the system such as $N_h = \int_0^\infty dx n_{1/g}(x)$. Since the statistics parameter $g$ is defined as the ratio between the total numbers of holes and particles, we have $g = N_h/N_p$ for $0 < g < 1$ (if $g > 1$ then the role of particles and holes is interchanged so that $g = N_p/N_h$). This means that the total charge of particles and holes in the system is zero so that the system is neutral from zero temperature to a finite temperature. Therefore, we must have Eq. (108). Thus, we claim that the result of $a_g(1) = 0$ comes from the definition of the statistics parameter.

However, a direct proof of the above result by explicitly calculating the generalized Riemann zeta functions is hard to do. Because when we numerically calculate the series expansions a finite term approximation may always yield a false result. This will be an interesting mathematical problem.

On the other hand, for higher moment terms:

$$I_s = (kT)^s f^{(s-1)}(\mu) \int_0^\infty dx x^{s-1} \{ n_g(x) + (-g)^{s-2} n_{1/g}(x) \}$$

$$= a_g(s) (kT)^s f^{(s-1)}(\mu),$$

(110)

there is no reason that the coefficients vanish. Hence, we are finally led to the following expansion:

$$I = \frac{1}{g} \int_0^\infty dx f(x) + \sum_{s=2}^{\infty} (kT)^s a_g(s) f^{(s-1)}(\mu).$$

(111)
4.3. Introduction of a generalized Riemann zeta function

The above type of an infinite series of Eq. (106) is known as a Dirichlet series.\textsuperscript{35} If we set as $\chi_l(s) = 1/l$ for all $l$, then $a_s$ becomes the Riemann zeta function, $\zeta(s)$ [see Eq. (55)]. Therefore, Eq. (106) is regarded as a generalization of the Riemann zeta function. However, the explicit expression of $\chi_l(s)$ is irrelevant when $g = 0$, since the second term diverges in this limit. Therefore, it had better define another generalized Riemann zeta function in the following form:

$$\zeta_g(s) = \sum_{l=1}^{\infty} \frac{a_l(g)}{l^s},$$  \hspace{1cm} (112)

where $a_l(g)$ is defined by Eq. (77). This yields

$$\zeta_{g=1}(s) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^s} = (1 - 2^{1-s})\zeta(s),$$  \hspace{1cm} (113)

$$\zeta_{g=0}(s) = \sum_{l=1}^{\infty} \frac{1}{l^s} = \zeta(s).$$  \hspace{1cm} (114)

Thus, $\zeta_g(s)$ of Eq. (112) is a generalization of the Riemann zeta function, $\zeta(s)$. Using Eq. (112) for Eqs. (106) and (107), Eq. (106) becomes

$$a_g(s) = \zeta_g(s) + (-g)^{s-2}\zeta_{1/g}(s).$$  \hspace{1cm} (115)

Let us now note the following: First, using the definition of the above generalized Riemann zeta functions, the relation Eq. (108) leads us to the following property of the generalized Riemann zeta function:

$$\zeta_g(1) = \frac{1}{g}\zeta_{1/g}(1).$$  \hspace{1cm} (116)

Second, from Eq. (115) we have that

$$a_g(2) = \zeta_g(2) + \zeta_{1/g}(2).$$  \hspace{1cm} (117)

Therefore, we have a symmetry property:

$$a_g(2) = a_{1/g}(2),$$  \hspace{1cm} (118)

from which we find that $a_g(2)$ is symmetric around $g = 1$.

Moreover, we note the following: The high temperature and low density expansions such as Eqs. (87) and (88) are also related to the generalized Riemann zeta
Generalized Sommerfeld Theory: Specific Heat of...

function. If we define

\[ \zeta_g(s, z) = \sum_{l=1}^{\infty} \frac{c_l(g)}{\lambda_{s}^{l-1}z}, \]  

then we have

\[ \frac{P}{kT} = \frac{2E/V}{D} \frac{1}{kT} = \frac{1}{\lambda D} \zeta_g(D/2 + 1, z), \]  

\[ \frac{N}{V} = \frac{1}{\lambda D} \zeta_g(D/2, z). \]  

Thus, when \( z = \exp(\beta\mu) = 1 \) (i.e. \( \mu = 0 \)), we get \( \zeta_g(s, 1) = \zeta_g(s) \).

4.4. Specific heat of a degenerate ideal g-on gas

Let us apply the formula for the low temperature and high density expansion of Eq. (116) to the energy and the density. For the energy \( E/V \), \( f(x) \) is regarded as \( xN_D(x) \), while for the density \( N/V \), \( f(x) \) is regarded as \( N_D(x) \). So, using the DOS of Eq. (22), we have \( N_D(x) = (m/2\pi\hbar^2)^{D/2}(1/\Gamma(D/2))x^{(D-2)/2} \) and \( xN_D(x) = (m/2\pi\hbar^2)^{D/2}(1/\Gamma(D/2))x^{D/2} \). We then have

\[ \frac{E}{V} = \frac{1}{g} \int_0^\mu dx xN_D(x) + \sum_{s=2}^{\infty} (kT)^s a_g(s)(xN_D)^{(s-1)}(\mu) \]

\[ = \frac{D}{2} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \left[ \frac{1}{g \Gamma(D/2 + 2)} \mu^{D/2+1} \right. \]

\[ + \sum_{s=2}^{\infty} (kT)^s a_g(s) \frac{1}{\Gamma(D/2 + 2 - s) \mu^{D/2+1-s}} \]

\[ = \frac{D}{2} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \left( \frac{1}{g \Gamma(D/2 + 2)} \mu^{D/2+1} \right) \]

\[ \times \left[ 1 + g \sum_{s=2}^{\infty} a_g(s) \frac{\Gamma(D/2 + 2)}{\Gamma(D/2 + 2 - s)} \left( \frac{kT}{\mu} \right)^s \right] \]

\[ = \frac{D}{2} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \left( \frac{1}{g \Gamma(D/2 + 2)} \mu^{D/2+1} \right) \]

\[ \times \left[ 1 + ga_g(2) \frac{D}{2} \left( \frac{D}{2} + 1 \right) \left( \frac{kT}{\mu} \right)^2 + \cdots \right], \]  

\[ (122) \]
\[
\frac{N}{V} = \frac{1}{g} \int_0^\mu dx N_D(x) + \sum_{s=2}^\infty (kT)^s a_g(s) N_D^{(s)}(\mu)
\]
\[
= \frac{m}{(2\pi\hbar^2)^{D/2}} \left[ \frac{1}{g} \frac{1}{\Gamma(D/2 + 1)} \frac{\mu^{D/2}}{\mu^{D/2}} \right]
\]
\[
+ \sum_{s=2}^\infty (kT)^s a_g(s) \frac{1}{\Gamma(D/2 + 1 - s)} \mu^{D/2-s}
\]
\[
= \frac{m}{(2\pi\hbar^2)^{D/2}} \left[ \frac{1}{g} \frac{1}{\Gamma(D/2 + 1)} \frac{\mu^{D/2}}{\mu^{D/2}} \right]
\]
\[
\times \left[ 1 + g \sum_{s=2}^\infty a_g(s) \frac{\Gamma(D/2 + 1)}{\Gamma(D/2 + 1 - s)} \left( \frac{kT}{\mu} \right)^s \right]
\]
\[
= \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \frac{1}{g} \frac{1}{\Gamma(D/2 + 1)} \frac{\mu^{D/2}}{\mu^{D/2}}
\]
\[
\times \left[ 1 + g a_g(2) \frac{D}{2} \left( \frac{D}{2} - 1 \right) \left( \frac{kT}{\mu} \right)^2 + \ldots \right].
\]
(123)

At \( T = 0 \) the chemical potential becomes the Fermi energy as \( \mu = \varepsilon_F \) and since the Wu’s distribution function vanishes (i.e. \( W_p = 0 \)) from Eq. (26) such that \( \langle n_p \rangle = 1/(W_p + g) = 1/g \), we then have

\[
\frac{N}{V} = \frac{1}{V} \sum_p \langle n_p \rangle = \frac{1}{g} \frac{1}{(2\pi\hbar)^D} \int_0^{\varepsilon_F} d^D p
\]
\[
= \frac{1}{g} \int_0^{\varepsilon_F} dx N_D(x) = \frac{1}{g} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2 + 1)} \frac{\varepsilon_F^{D/2}}{\varepsilon_F^{D/2}}
\]
(124)

\[
\frac{E}{V} = \frac{1}{V} \sum_p \varepsilon_p \langle n_p \rangle = \frac{1}{g} \frac{1}{(2\pi\hbar)^D} \int_0^{\varepsilon_F} d^D p \frac{p^2}{2m}
\]
\[
= \frac{1}{g} \int_0^{\varepsilon_F} dx x N_D(x) = \frac{D}{2} \frac{1}{g} \left( \frac{m}{2\pi\hbar^2} \right)^{D/2} \frac{1}{\Gamma(D/2 + 2)} \frac{\varepsilon_F^{D/2+1}}{\varepsilon_F^{D/2+1}}
\]
\[
= \frac{D}{D + 2} \frac{N}{V} \varepsilon_F.
\]
(125)

Comparing Eqs. (124) and (125) with Eqs. (63) and (64), these equations are equivalent to each other apart from the factor \( 1/g \) in this case. This factor comes from the degeneracy of the ground state of an ideal \( g \)-on gas. Substituting Eq. (124) into
the left hand side of Eq. (123), we can rewrite it as

\[
\varepsilon_F^{D/2} = \mu^{D/2} \left[ 1 + g \sum_{s=2}^{\infty} a_g(s) \frac{\Gamma(D/2 + 1 + s)}{\Gamma(D/2 + 1 - s)} \left( \frac{kT}{\mu} \right)^s \right]
\]

\[
= \mu^{D/2} \left[ 1 + ga_g(2) \frac{D}{2} \left( \frac{D}{2} - 1 \right) \left( \frac{kT}{\mu} \right)^2 + \cdots \right]. \quad (126)
\]

Let us invert the above for \( \mu \) in terms of \( \varepsilon_F \). In this case, the expansion consists of odd order terms as well as even order terms for more than the second order with respect to \( kT/\mu \). Here the odd order terms come from the particle–hole asymmetry, and so we cannot use the expansions in the Fermi gas case where the expansions consist of only the powers of even order with respect to \( kT/\mu \). Thus, there appears a delicate problem to take an expansion up to the fourth order of \( kT/\mu \). However, since there is no linear term of \( kT/\mu \), it is easy to expand up to the second order as follows.

Suppose that the chemical potential \( \mu \) is expanded up to the second order as

\[
\mu = \varepsilon_F (1 + x \alpha^2), \quad \alpha = \frac{kT}{\varepsilon_F}. \quad (127)
\]

From this, we find the following

\[
\mu^{-1} = \varepsilon_F^{-1} (1 - x \alpha^2), \quad (128)
\]

\[
\mu^{-2} = \varepsilon_F^{-2} (1 - 2x \alpha^2). \quad (129)
\]

Substituting Eq. (127) into \( \mu^{D/2} \), we obtain

\[
\mu^{D/2} = \varepsilon_F^{D/2} \left[ 1 + \frac{D}{2} x \alpha^2 \right]. \quad (130)
\]

Substituting Eqs. (128)–(130) into Eq. (126) and comparing both sides, we find \( x \) as

\[
x = -g \left( \frac{D}{2} - 1 \right) a_g(2). \quad (131)
\]

Hence, by substitution we get

\[
\mu = \varepsilon_F \left[ 1 - g \left( \frac{D}{2} - 1 \right) a_g(2) \left( \frac{kT}{\varepsilon_F} \right)^2 + \cdots \right]. \quad (132)
\]
Using this for $\mu^{D/2+1}$, we obtain

$$\mu^{D/2+1} = \varepsilon_p^{D/2+1} \left[ 1 - g \left( \frac{D}{2} + 1 \right) \left( \frac{D}{2} - 1 \right) a_g(2) \alpha^2 \right]. \quad (133)$$

Similarly we get

$$1 + g a_g(2) \frac{D}{2} \left( \frac{D}{2} + 1 \right) \left( \frac{kT}{\mu} \right)^2 + \cdots = 1 + g \left( \frac{D}{2} + 1 \right) \frac{D}{2} a_g(2) \alpha^2 + \cdots. \quad (134)$$

Substituting Eqs. (133) and (134) into Eq. (125), the total energy is expanded as

$$\frac{E}{V} = \frac{D}{D+2} N \varepsilon_p \left[ 1 + g \left( \frac{D}{2} + 1 \right) a_g(2) \left( \frac{kT}{\varepsilon_p} \right)^2 + O(T^3) \right]. \quad (135)$$

From the above, we finally find the specific heat of a degenerate gas in the following:

$$C_v = \left( \frac{\partial E}{\partial T} \right)_V = \frac{D}{D+2} N \varepsilon_p \left[ 2 g a_g(2) \left( \frac{D}{2} + 1 \right) \left( \frac{k}{\varepsilon_p} \right)^2 T + O(T^2) \right]$$

$$= D N k g a_g(2) \left( \frac{k}{\varepsilon_p} \right)^2 + O(T^2). \quad (136)$$

In this way, we find the linear dependent specific heat on temperature for a degenerate ideal gas, where the coefficient depends on only statistics $g$ and dimensionality $D$ of the system. For if $g = 1$, then the particle–hole symmetry is recovered such that $a_g(s)$ with $s$ odd vanish and $a_g(s)$ with $s$ even become $a_g(2s) = 2(1-2^{-2s})\zeta(2s)$. Therefore, the above specific heat coincides with that of an ideal Fermi gas [Eq. (68)], since $g a_g(2) = \zeta(2) = \pi^2/6$ if $g = 1$. This is one of our main results of this paper.

5. Discussion

It has been believed that in the strict sense the FES is realized only in one dimension.\textsuperscript{5,13,14} For example, the Hilbert space in the LLL of the fractional quantum Hall system is reduced to give the one-dimensional spectrum with high degeneracy so that the problem becomes one-dimensional,\textsuperscript{8} and the CSM is trivially a one-dimensional system providing the one-dimensional spectrum with quadratic dispersion as well.\textsuperscript{5–7} Therefore, many studies have been done on this context.\textsuperscript{1–27}

On the other hand, in the previous sections we have assumed that the spectrum of a free gas in higher dimensions is parabolic with $\varepsilon_p = p^2/2m$, apart from the above restricted context. However, even in this case, this assumption is not always true in a solid since the spectrum is influenced by the presence of periodic lattice
structure. This is one of the main failures of the Sommerfeld theory in real solids. Therefore, to escape from the same failure, we must take into account the effect of lattice periodicity on the energy spectrum. So, let us generalize the low temperature and high density expansions to those for the system with a more realistic energy spectrum.

To do so, let us assume that the DOS of the system is given to include all information of the lattice structure. Denote it by $N_D(\varepsilon)$. Then, using our low temperature and high density expansion formula of Eq. (115), the total energy and the number of particles are given by

$$
n = \frac{N}{V} = \frac{1}{g} \int_{0}^{\mu} dx N_D(x) + \sum_{s=2}^{\infty} \langle kT \rangle^s a_g(s) N_D^{(s-1)}(\mu)
$$

$$
= \frac{1}{g} \int_{0}^{\mu} dx N_D(x) + \langle kT \rangle^2 a_g(2) N_D^2(\mu) + \cdots, \quad (137)
$$

$$
E = \frac{1}{V} = \frac{1}{g} \int_{0}^{\mu} dx x N_D(x) + \sum_{s=2}^{\infty} \langle kT \rangle^s a_g(s) x N_D^{(s-1)}(\mu)
$$

$$
= \frac{1}{g} \int_{0}^{\mu} dx x N_D(x) + \langle kT \rangle^2 a_g(2) [\mu N_D^2(\mu) + N_D(\mu)] + \cdots. \quad (138)
$$

Now, we follow the argument of Ashcroft and Mermin as follows: Using Taylor expansions of $N_D(\mu)$, $N_D'(\mu)$, $\int_{0}^{\mu} dx N_D(x)$, and $\int_{0}^{\mu} dx x N_D(x)$ around $\mu = \varepsilon_F$ up to the second order of $\mu - \varepsilon_F$, we find that

$$
N_D(\mu) = N_D(\varepsilon_F) + (\mu - \varepsilon_F) N_D'(\varepsilon_F) + \frac{1}{2} (\mu - \varepsilon_F)^2 N_D''(\varepsilon_F) + \cdots, \quad (139)
$$

$$
N_D'(\mu) = N_D'(\varepsilon_F) + (\mu - \varepsilon_F) N_D''(\varepsilon_F) + \frac{1}{2} (\mu - \varepsilon_F)^2 N_D'''(\varepsilon_F) + \cdots, \quad (140)
$$

$$
\int_{0}^{\mu} dx N_D(x) = \int_{0}^{\varepsilon_F} dx N_D(x) + (\mu - \varepsilon_F) N_D(\varepsilon_F)
$$

$$
+ \frac{1}{2} (\mu - \varepsilon_F)^2 N_D'(\varepsilon_F) + \cdots, \quad (141)
$$

$$
\int_{0}^{\mu} dx x N_D(x) = \int_{0}^{\varepsilon_F} dx x N_D(x) + (\mu - \varepsilon_F) \varepsilon_F N_D(\varepsilon_F)
$$

$$
+ \frac{1}{2} (\mu - \varepsilon_F)^2 \{\varepsilon_F N_D'(\varepsilon_F) + N_D(\varepsilon_F)\} + \cdots. \quad (142)
$$
Substituting Eqs. (139)–(141) into Eq. (137), we get

\[ n = n_0 + (kT)^2 a_g(2) N'_D(\varepsilon_F) \]
\[ + (\mu - \varepsilon_F) \left\{ \frac{1}{g} N_D(\varepsilon_F) + (kT)^2 a_g(2) N''_D(\varepsilon_F) \right\} \]
\[ + \frac{1}{2} (\mu - \varepsilon_F)^2 \left\{ \frac{1}{g} N'_D(\varepsilon_F) + (kT)^2 a_g(2) N'''_D(\varepsilon_F) \right\} + \cdots, \]

(143)

where \( n_0 \equiv (1/g) \int_{0}^{\varepsilon_F} dx N_D(x) \). Let us suppose the expansion of \( \mu \) in the form of Eq. (124). Substituting it into Eq. (143), we get

\[ n = n_0 + (kT)^2 \left\{ \frac{\varepsilon_D(\varepsilon_F)}{\varepsilon_F} + a_g(2) N'_D(\varepsilon_F) \right\} + \cdots. \]

(144)

Since \( n = n_0 \) for all temperature, we obtain the value of \( x \) as

\[ x = -g a_g(2) \varepsilon_F \frac{N'_D(\varepsilon_F)}{N_D(\varepsilon_F)}. \]

(145)

Hence, we find that the expansion of \( \mu \) in the following:

\[ \mu = \varepsilon_F - g a_g(2) \varepsilon_F \frac{N'_D(\varepsilon_F)}{N_D(\varepsilon_F)} \left( \frac{kT}{\varepsilon_F} \right)^2 + \cdots, \]

(146)

which is the generalization of Eq. (132).

Next, substituting Eqs. (139)–(141) into Eq. (138) together with \( \mu = \varepsilon_F + (\mu - \varepsilon_F) \), we similarly find that

\[ \frac{E}{V} = \frac{1}{g} \int_{0}^{\varepsilon_F} dx x N_D(x) + (kT)^2 a_g(2) [\varepsilon_F N'_D(\varepsilon_F) + N_D(\varepsilon_F)] \]
\[ + (\mu - \varepsilon_F) \left\{ \frac{1}{g} \varepsilon_F N_D(\varepsilon_F) + (kT)^2 a_g(2) [\varepsilon_F N'_D(\varepsilon_F) + 2N''_D(\varepsilon_F)] \right\} \]
\[ + \frac{1}{2} (\mu - \varepsilon_F)^2 \left\{ \frac{1}{g} \varepsilon_F N'_D(\varepsilon_F) + N_D(\varepsilon_F) \right\} \]
\[ + (kT)^2 a_g(2) [\varepsilon_F N'''_D(\varepsilon_F) + 3N''_D(\varepsilon_F)] \right\} + \cdots, \]

(147)

where \( E/V \equiv (1/g) \int_{0}^{\varepsilon_F} dx x N_D(x) \). Using Eq. (146) in Eq. (147), we finally obtain the following simple expression:

\[ \frac{E}{V} = \frac{E_0}{V} + (kT)^2 a_g(2) N_D(\varepsilon_F) + O(T^3). \]

(148)
Hence, differentiation of Eq. (148) with respect to $T$ yields

$$C_v = 2a_v(2)k^2 N_D(\epsilon_F)T + O(T^2).$$

(149)

The above formulas are the generalization of Eqs. (135) and (136), respectively. Indeed, if we use the DOS of Eq. (22) in Eq. (149), then we find that $N_D(\epsilon_F) = (m/2\pi \hbar^2)^{D/2} |1/T(D/2)|^{(D-2)/2} g(D/2\epsilon_F)/(N/V)$, and therefore we find the same expression as Eq. (133), once again. In this way, the above formula of Eq. (149) is the most general expression for the specific heat of a degenerate $g$-on gas in a $D$-dimensional periodic lattice structure.

6. Conclusion

In conclusion we have discussed the way of calculating the specific heat of a degenerate $g$-on gas in $D$ spatial dimensions. We have developed a method of low temperature and high density expansions for a degenerate $g$-on gas with $g > 0$, which can be regarded as a generalization of the Sommerfeld expansions for a degenerate Fermi gas in a metal, as well as the high temperature and low density expansions known as virial expansions. In both expansions we have shown that the cluster expansion [Eq. (76)] for the CSM in one dimension plays a very important role in the theory: In the latter we have used it to represent the virial expansions. On the other hand, in the former we have used it to generalize the Riemann zeta function to that with statistics parameter $g$. By applying the generalized Sommerfeld expansions, we have obtained the formulas for specific heat of both an ideal $g$-on gas and a $g$-on gas in a periodic lattice structure in $D$ dimensions. We have found that the specific heat of the $g$-on gases for $g > 0$ is $T$-linear at very low temperature, and the coefficient depends on only statistics parameter $g$ and dimensionality $D$ of the system. In this way, the theory that we have presented in the present paper can be thought of as a generalized Sommerfeld theory for the degenerate $g$-on gases. Finally, we would like to expect that the effect of statistics $g$ can be detected by a testable experiment such as a measurement of specific heat. This claim is true even for the CSM in one dimension as well, since our formula for the specific heat of a $g$-on gas is applicable to any dimension. The study on this point would be very interesting for foundations of condensed matter physics as well as QSM.

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References