Theory of quasiperiodic lattices. II. Generic trace map and invariant surface

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The generic trace map is established for binary quasiperiodic lattices represented by any irrational number. This is a generalization of the renormalization-group method of Kohmoto, Kadanoff, and Tang (KKT) for the Fibonacci lattice. All the trace maps for any quasiperiodic lattice preserve the same invariant surface first discovered by KKT. There is a special set of points on the invariant surface, which we call the invariant six-cycle. These are fixed points of the trace maps. We are able to obtain representations of the trace maps or scaling transformations on the invariant six-cycle. This enables us to determine the periods of the trace maps, which are very important in order to know the scaling property of the electronic wave function and energy spectrum at the band center.

I. INTRODUCTION

In physics, the existence of the trace map and its invariant surface was first found by Kohmoto, Kadanoff, and Tang (KKT) for the Fibonacci lattice.[1] The important role of the invariant surface in understanding the Cantor-set-type band spectrum of the Schrödinger equation of a Fibonacci lattice has been described by Kohmoto and Oono,[2] Kohmoto and Banavar,[3] and Kohmoto, Sutherland, and Tang.[4] There is a triple of traces (x,y,z); $x=\frac{1}{2}\mathrm{Tr}(B)$, $y=\frac{1}{2}\mathrm{Tr}(A)$, and $z=\frac{1}{2}\mathrm{Tr}(AB)$, that preserve the invariant surface

$$\underline{I}=x^2+y^2+z^2-2xyz-1 . \tag{1}$$

This invariant-surface structure was first presented by Kohmoto and Oono.[2] (See Fig. 1.) There have been many applications of the KKT technique for the Fibonacci lattice to other physical systems.[5-8] Recently, generalizations of the trace map to other quasiperiodic lattices were made by Gumps and Ali[9] and Holzer.[10] To date, most types of quasiperiodic lattices are not well understood.

Historically speaking, and amazingly, the existences of the trace map and its invariant surface were already known by Fricke and Klein who were mathematicians in the previous century. At that time there were no physical applications for them[11] because it preceeded the discovery of quantum mechanics. More recently, Cohn used them to solve the Markoff spectrum problem.[12-18] This was the first systematical application of the work of Fricke and Klein. Rosenberger also studied some mathematical properties of the trace map and the invariant surface.[19-22] However, relationships between the trace map, the invariant surface, and quasiperiodic lattices have not been understood until recently. We have discovered these relationships,[23,24] which will be discussed in this paper.

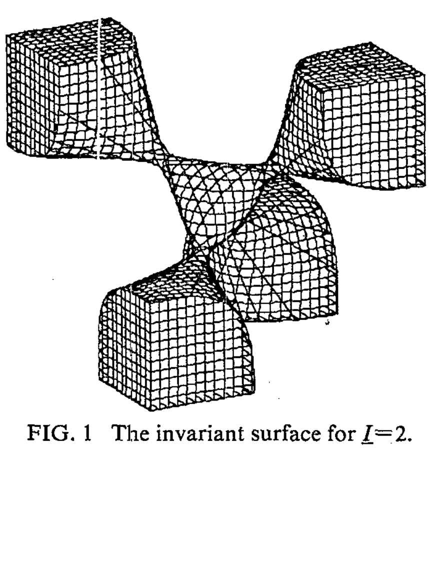

FIG. 1 The invariant surface for $\underline{I}=2$.

In the preceding paper[24] we discussed a group of scaling transformations $\underline{S}$ (called the *scaling group*) used to construct quasiperiodic lattices. The group of scaling transformations $\underline{S}$ is the following: $\mathcal{R}(A,B)=(A,AB)$, right multiplication; $\mathcal{L}(A,B)=(AB,B)$, left multiplication; and $\underline{X}(A,B)=(B,A)$, exchange. Since the scaling transformations $\underline{S}$ are constructed from the generators of a *substitution group* $\underline{F}_1$,[24] the scaling transformations $\underline{S}$ induce trace maps according to the scaling transformations, preserving the invariant surface defined by (1). This will be discussed in the first part of this paper. The invariant surface has special points which we call the invariant six-cycle of the trace map. These fixed points of the trace maps, which were first found by Kohmoto and Oono, play an important role in determining the scaling exponent of the electron energy spectrum and wave function at the band center.[2,4]

II. THE GENERIC TRACE MAP AND INVARIANT SURFACE

In the preceding paper,[24] we have discussed a group of scaling transformations in terms of the generators of a substitution group, which is a subgroup of a free group made by two matrices A and B belonging to $\underline{G}=SL(2,\underline{C})$. We shall call (A,B) the seed for a quasiperiodic lattice.

As we have seen, the scaling transformations, or the inflation-deflation transformations, of quasiperiodic lattices are of the form

$$\underline{X}\mathcal{R}^n(A,B)=(A^nB,A),\quad \underline{X}\mathcal{L}^n(B,A)=(A,BA^n)\ , \tag{2}$$

where we have used elementary transformations of $\underline{S}$, $\mathcal{R}$, $\mathcal{L}$, and $\underline{X}$. We define $\underline{X}\mathcal{L}^n \equiv \underline{U}_n$, $\underline{X}\mathcal{R}^n \equiv \underline{V}_n$.

We also define a commutator matrix $K \equiv B^{-1}A^{-1}BA$ such that

$$\Lambda=\mathrm{Tr}(K)=\mathrm{Tr}(B^{-1}A^{-1}BA)=\mathrm{Tr}(BAB^{-1}A^{-1})\ . \tag{3}$$

We further assume that there exists a matrix U such that $A^{-1}=UAU^{-1}$ for all A in $\underline{G}$. Then $\mathrm{Tr}(A)=\mathrm{Tr}(A^{-1})$, and thus $\Lambda=\mathrm{Tr}(ABA^{-1}B^{-1})$. We see that Λ is invariant under $\underline{X}$ and $\mathcal{R}$ (or $\mathcal{L}$):

$$\underline{X}\Lambda=\mathrm{Tr}(A^{-1}B^{-1}AB)=\Lambda\ ,$$

$$\mathcal{R}\Lambda=\mathrm{Tr}(B^{-1}A^{-1}A^{-1}ABA)=\Lambda\ ,$$

and

$$\mathcal{L}\Lambda=\mathrm{Tr}(B^{-1}B^{-1}A^{-1}BAB)=\Lambda\ .$$

Thus, $\underline{X}$ and $\mathcal{L}$ (or $\mathcal{R}$) generate the group $\underline{S}$ which acts on $\underline{G}\times\underline{G}$, and includes the inflation-deflation transformations $\underline{U}_n$ and $\underline{V}_n$. Λ is invariant under $\underline{S}$, which is related to Eq. (1) through $\underline{I}=(\Lambda-2)/4$.

A very important property of the scaling group $\underline{S}$ is that $\underline{S}$ commutes with the inner automorphisms of $\underline{G}$, since, if g is any element of $\underline{G}$ and t is an element of $\underline{S}$, then

$$t[g(A,B)g^{-1}]=g[t(A,B)]g^{-1}\ . \tag{4}$$

This implies that $\underline{S}$ induces a symmetry group $\underline{S}'$ called the trace map on the invariants (A,B) under inner automorphisms.

Now we define the triple

$$(x,y,z)=(\tfrac{1}{2}\mathrm{Tr}(B),\tfrac{1}{2}\mathrm{Tr}(A),\tfrac{1}{2}\mathrm{Tr}(AB))\ .$$

Then the inner automorphisms induce the following transformations:

$$\begin{aligned}&\underline{I}(x,y,z)=(x,y,z)\ ,\\ &\underline{X}(x,y,z)=(y,x,z)\ ,\\ &\mathcal{L}(x,y,z)=(z,y,2yz-x)\ ,\\ &\mathcal{R}(x,y,z)=(x,z,2xz-y)\ .\end{aligned} \tag{5}$$

Here we have used the Fricke identity to prove Eq. (5). It is easy to see that these transformations preserve the invariant surface (1).

Now the inflation-deflation transformations of Eq. (2) induce the following trace maps:

$$\begin{aligned}&\underline{U}_n(x,y,z)\equiv\underline{X}\mathcal{L}^n(x,y,z)=(y,C_{n-1}(y)z-C_{n-2}(y)x,C_n(y)z-C_{n-1}(y)x)\ ,\\ &\underline{V}_n(x,y,z)\equiv\underline{X}\mathcal{R}^n(x,y,z)=(C_{n-1}(x)z-C_{n-2}(x)y,x,C_n(x)z-C_{n-1}(x)y)\ ,\end{aligned} \tag{6}$$

where we have used the Chebyshev polynomials of the second kind $C_n(x)$, which are defined by $C_{-1}(x)=0$, $C_0(x)=1$, and

$$C_n(x)=2xC_{n-1}(x)-C_{n-2}(x)\ .$$

It is not difficult to directly prove that Eq. (6) preserves the invariant surface (1) if we notice that the Chebyshev polynomials of the second kind enjoy the following properties:

$$C_n^2-C_{n-1}C_{n+1}=1\ ,$$

$$C_n^2+C_{n-1}^2-2xC_nC_{n-1}=1\ .$$

From the scaling transformations of the quasiperiodic lattice from the seed of (x,y,z), we induce the scaling transformations on the triple (x,y,z):

$$\begin{aligned}&(x_L,y_L,z_L)=\underline{U}_{n_k}\underline{U}_{n_{k-1}}\cdots\underline{U}_{n_0}(x,y,z)\ ,\\ &(x_R,y_R,z_R)=\underline{V}_{n_k}\underline{V}_{n_{k-1}}\cdots V_{n_0}(x,y,z)\ .\end{aligned} \tag{7}$$

These are dynamical maps of traces on the invariant surface (1), according to the kth-order approximation of the irrational numbers λ_k, having the tail $[n_0,n_1,\ldots,n_k]$.

We now place some previously studied quasiperiodic systems in the context of this theory. When we take $n_0=n_1=\cdots=n_k=1$, this corresponds to a Fibonacci lattice. Equation (7) yields the same result as the KKT renormalization-group method. If we take $n_0=n_1=\cdots=n_k=2$, it gives the scaling transformation for a silver-mean lattice, if $n_0=n_1=\cdots=n_k=3$, it gives the scaling transformation for a bronze-mean lattice. If

$$n_0=n_2=\cdots=n_{2m}=1\ ,$$

$$n_1=n_3=\cdots=n_{2m+1}=2\ ,$$

Eq. (7) gives the scaling transformation for a gold-silver alloy-mean lattice. Once we take a tail of any irrational number, we are able to determine the scaling transformation. This theory is the generalization of the trace map of the Fibonacci lattice to any quasiperiodic lattice.

III. THE INVARIANT SIX-CYCLE OF THE TRACE MAP

We are concerned with a particular aspect of the invariant surface which is defined by Eq. (1). This particular and important set of points on the invariant surface was first found by Kohmoto and Oono[2] while investigating the trace map of the Fibonacci lattice. This set is called the invariant six-cycle. (See Fig. 2.)

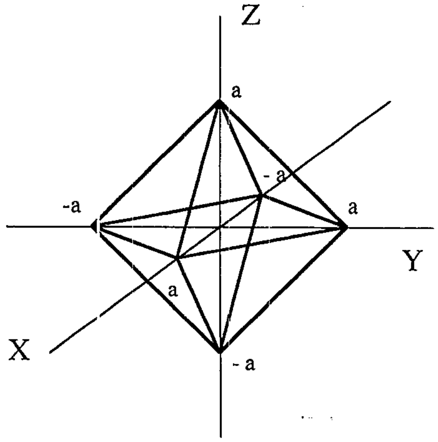

FIG. 2. The invariant six-cycle. This forms the symmetry of a tetrahedron.

We look at the set of triples which is defined by the following:

$$(a,0,0)=\psi_1,\quad (-a,0,0)=-\psi_1\ ,$$
$$(0,a,0)=\psi_2,\quad (0,-a,0)=-\psi_2\ ,$$
$$(0,0,a)=\psi_3,\quad (0,0,-a)=-\psi_3\ ,$$

where a is related to the invariant Λ by $a=\sqrt{\Lambda+2}/2$.

Under the actions of trace maps, we can determine the representations of the generators $\underline{X}$, $\mathcal{L}$ (or $\mathcal{R}$) on the invariant six-cycle. First, we see the action of the exchange $\underline{X}$ is

$$\underline{X}\psi_1=\psi_2,\quad \underline{X}\psi_2=\psi_1,\quad \underline{X}\psi_3=\psi_3\ , \tag{8}$$

and so the matrix representation of $\underline{X}$ is

$$\underline{D}(\underline{X})=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\underline{D}(\underline{X})^t,\quad \det\underline{D}(\underline{X})=-1\ . \tag{9}$$

Likewise, for the left multiplication $\mathcal{L}$, the action is

$$\mathcal{L}\psi_1=-\psi_3,\quad \mathcal{L}\psi_2=\psi_2,\quad \mathcal{L}\psi_3=\psi_1\ , \tag{10}$$

and hence the matrix representation of $\mathcal{L}$ is

$$\underline{D}(\mathcal{L})=\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\ ,$$
$$\underline{D}(\mathcal{L})^t=\underline{D}(\mathcal{L})^{-1}\ , \tag{11}$$
$$\det\underline{D}(\mathcal{L})=1$$

Similarly, for the right multiplication $\mathcal{R}$, the action is

$$\mathcal{R}\psi_1=\psi_1,\quad \mathcal{R}\psi_2=-\psi_3,\quad \mathcal{R}\psi_3=\psi_2\ , \tag{12}$$

and the representation of $\mathcal{R}$ is

$$\underline{D}(\mathcal{R})=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\ ,$$
$$\underline{D}(\mathcal{R})^t=\underline{D}(\mathcal{R})^{-1}\ , \tag{13}$$
$$\det\underline{D}(\mathcal{R})=1\ .$$

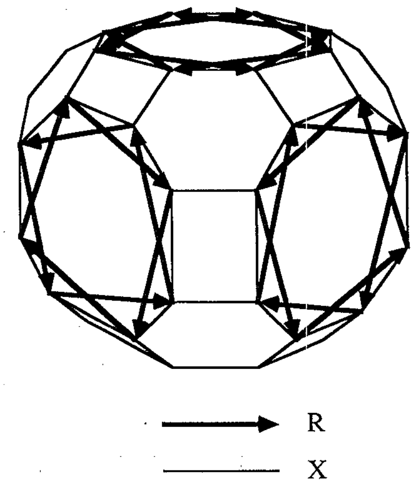

FIG. 3. The full group symmetry of an octahedron. This group is generated by $\underline{X}$, the reflection about the $x=y$ plane and by $\mathcal{R}$ ($\mathcal{L}$) the rotation about the x axis (y axis) by $\pi/2$. The order of the group is 48.

It is easy to see that this group has all the group symmetries of the octahedron, including both reflections and rotations. The order of this group is 48. $\underline{X}$ is a reflection about the $x=y$ plane, and $\mathcal{R}$ ($\mathcal{L}$) is a rotation about the x axis (y axis) by $\pi/2$. This group structure is shown in Fig. 3. As a result of this symmetry, the trace map of the invariant six-cycle hops about from one group element to another, with dynamics determined by the continued fraction expansion of λ—the ratio of A's to B's in the sequence. Finally, we note that

$$\underline{D}(\underline{X})^2=\underline{D}(\mathcal{L})^4=\underline{D}(\mathcal{R})^4=1\ . \tag{14}$$

IV. REPRESENTATIONS OF THE TRACE MAPS ON THE INVARIANT SIX-CYCLE AND THEIR PERIODS

The representation of the generators $\underline{X}$, $\mathcal{L}$, and $\mathcal{R}$ plays a crucially important role in finding the periods of the trace map on the invariant set, namely the invariant six-cycle. If we start with one of these points on the invariant surface, under the scaling transformations, it never escapes from points of the set of the invariant six-cycle. In other words, these points are invariants or fixed points of the trace map. Physically, this set corresponds to the center of each band determined by the Schrödinger equation for an electron on a quasiperiodic lattice.[2-4] This physical point is planned to be discussed elsewhere.[25]

We discuss how to determine the periods of the trace map in terms of the representation. Now we think of scaling transformations in the form of $\underline{U}_n\equiv\underline{X}\mathcal{L}^n$ or $\underline{V}_n\equiv\underline{X}\mathcal{R}^n$. From Eq. (14), the integer n can be measured by modulo 4. Therefore, we concentrate only on the following four cases: for the left multiplication, $\underline{U}_1,\underline{U}_2,\underline{U}_3,\underline{U}_4=\underline{X}=\underline{U}_0$, and the same for right multiplication, replacing $\mathcal{L}$ with $\mathcal{R}$.

Since we know that the actions of $\mathcal{L}$ and $\mathcal{R}$ are

$$\begin{aligned}&\mathcal{L}\psi_1=-\psi_3,\quad \mathcal{L}\psi_2=\psi_2,\quad \mathcal{L}\psi_3=\psi_1\ ,\\&\mathcal{R}\psi_1=\psi_1,\quad \mathcal{R}\psi_2=-\psi_3,\quad \mathcal{R}\psi_3=\psi_2\ ,\end{aligned} \tag{15}$$

and hence the representations of $\mathcal{L}$ and $\mathcal{R}$ are

$$\underline{D}(\mathcal{L})=\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix},\quad \underline{D}(\mathcal{R})=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}. \tag{16}$$

Similarly, we have representations of $\mathcal{L}^2$, $\mathcal{R}^2$, $\mathcal{L}^3$, and $\mathcal{R}^3$ as

$$\underline{D}(\mathcal{L}^2)=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix},\quad \underline{D}(\mathcal{R}^2)=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},\quad \underline{D}(\mathcal{L}^3)=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix},\quad \underline{D}(\mathcal{R}^3)=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}. \tag{17}$$

From these we can determine the representations of $\underline{U}_1,\underline{U}_2,\underline{U}_3,\underline{U}_4=\underline{X}=\underline{U}_0$, (i.e., $\underline{X}\mathcal{L},\underline{X}\mathcal{L}^2,\underline{X}\mathcal{L}^3,\underline{X}\mathcal{L}^4=\underline{X}$) on the invariant six-cycle. First, we see that the action of $\underline{X}\mathcal{L}$ is

$$\underline{X}\mathcal{L}\psi_1=\psi_2,\quad \underline{X}\mathcal{L}\psi_2=-\psi_3,\quad \underline{X}\mathcal{L}\psi_3=\psi_1 \tag{18}$$

with the resultant representation of $\underline{X}\mathcal{L}$ as

$$\underline{D}(\underline{U}_1)=\underline{D}(\underline{X}\mathcal{L})\equiv\underline{D}(\mathcal{L})\underline{D}(\underline{X})=\begin{bmatrix}0&0&1\\1&0&0\\0&-1&0\end{bmatrix}. \tag{19}$$

Second, we see the action of $\underline{X}\mathcal{L}^2$ is

$$\underline{X}\mathcal{L}^2\psi_1=\psi_2,\quad \underline{X}\mathcal{L}^2\psi_2=-\psi_1,\quad \underline{X}\mathcal{L}^2\psi_3=-\psi_3\,, \tag{20}$$

with the resultant representation of $\underline{X}\mathcal{L}^2$ as

$$\underline{D}(\underline{U}_2)=\underline{D}(\underline{X}\mathcal{L}^2)\equiv\underline{D}(\mathcal{L}^2)\underline{D}(\underline{X})=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix}. \tag{21}$$

Third, we see the action of $\underline{X}\mathcal{L}^3$ is

$$\underline{X}\mathcal{L}^3\psi_1=\psi_2,\quad \underline{X}\mathcal{L}^3\psi_2=\psi_3,\quad \underline{X}\mathcal{L}^3\psi_3=-\psi_1\,, \tag{22}$$

with the resultant representation of $\underline{X}\mathcal{L}^3$ as

$$\underline{D}(\underline{U}_3)=\underline{D}(\underline{X}\mathcal{L}^3)\equiv\underline{D}(\mathcal{L}^3)\underline{D}(\underline{X})=\begin{bmatrix}0&0&-1\\1&0&0\\0&-1&0\end{bmatrix}. \tag{23}$$

Finally, we see that $\underline{D}(\underline{U}_4)=\underline{D}(\underline{U}_0)=\underline{D}(\underline{X})$.

From above results, it is easy to see the following properties of the representation:

$$\begin{aligned}[\underline{D}(\underline{X}\mathcal{L})]^6&=[\underline{D}(\underline{X}\mathcal{L}^2)]^4\\&=[\underline{D}(\underline{X}\mathcal{L}^3)]^6\\&=[\underline{D}(\underline{X}\mathcal{L}^4)]^2=1\,.\end{aligned} \tag{24}$$

The first expression $[\underline{D}(\underline{X}\mathcal{L})]^6$ corresponds to the Fibonacci lattice which has a period of six; the second, to the silver-mean lattice having a period of four; the third, to the bronze-mean lattice having a period of six; the last, to the aluminum-mean lattice (i.e., all $n_k=4$) having a period of two. We can clearly see the advantage of the representation of scaling transformations on the invariant six-cycle in order to determine the period of the scaling transformations at the band centers.

In general, if we have a string of $\underline{X}$ and $\mathcal{L}$ according to the continued fraction expansion of λ such that

$$(x_L,y_L,z_L)\equiv\lim_{k\to\infty}\underline{U}_{n_k}\underline{U}_{n_{k-1}}\cdots\underline{U}_{n_0}(x,y,z)\,, \tag{25}$$

then the trace map induces the representation of the string on the invariant six-cycle:

$$\begin{bmatrix}y_1'\\y_2'\\y_3'\end{bmatrix}=\lim_{k\to\infty}\underline{D}(\underline{U}_{n_k})\underline{D}(\underline{U}_{n_{k-1}})\cdots\underline{D}(\underline{U}_{n_0})\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}. \tag{26}$$

If the representation of the string of generators becomes the identity $\underline{1}$, in other words

$$[\underline{D}(\underline{U}_{n_k})\underline{D}(\underline{U}_{n_{k-1}})\cdots\underline{D}(\underline{U}_{n_0})]^p=\underline{1}\,, \tag{27}$$

then p is the period of the scaling transformations on the invariant six-cycle.

In conclusion, we have established the trace map for any quasiperiodic lattice. This trace map was induced from the scaling transformation of the transfer matrices for the quasiperiodic lattice. All the trace maps preserve the same invariant surface as that found by KKT, on which there is an invariant set of points called the invariant six-cycle. We have shown that the trace maps have three-dimensional representations on the invariant six-cycle. This allows us to determine periods of the trace maps on the invariant six-cycle. These periods are related to the scaling property of both the electronic energy spectrum and wave function at the center of band, as shown by Kohmoto and Oono,[2] and Kohmoto, Sutherland, and Tang.[4] This point is planned to be discussed elsewhere.[25] Very similar work has recently been published by Kalugin, Kitaev, and Levitov.[26]

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