Quantum Statistical Mechanics of an Ideal Gas with Fractional Exclusion Statistics in Arbitrary Dimensions

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The quantum statistical mechanics of an ideal gas with fractional exclusion (i.e., Haldane-Wu) statistics in arbitrary dimensions is discussed. The general formulation for pressure and density of the system is obtained in a closed form in terms of the D-dimensional momentum representation, which can be regarded as a natural generalization of the classic results for Fermi and Bose gases. Using this, it is shown that ideal gases with fractional exclusion statistics can be regarded as composites of fermions and bosons, and that no condensation occurs at low temperature except for the pure boson case. [S0031-9007(97)03012-3]

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The concept of fractional exclusion statistics (FES) has been of much interest in recent years [1–16]. Many investigations of FES have been done on low-dimensional systems such as the fractional quantum Hall system [17] and the Calogero-Sutherland model (CSM) [18,19], and the concept of FES played a very important role in these systems. However, in higher-dimensional systems, much work has not yet been made, except the work of Nayak and Wilczek [9]. The general formulation of quantum statistical mechanics (QSM) [20] that enables one to calculate thermal properties such as the equation of state for an ideal gas with FES is still lacking, although the concept of FES was given for arbitrary dimensional systems. In this Letter, I present a generalization of the method to an ideal gas with FES in arbitrary dimensions.

Haldane [1] first presented a generalized version of the Pauli principle using state counting methods. He defined the statistical interactions g_{ab} through the differential relation $\Delta D_a = -\sum_b g_{ab}\Delta N_b$, where D_a is the dimension of the Hilbert space H_a of states of a single particle of species a confined to a finite region of matter, and N_a is the number of particles of species a. D_a can change as particles are added, while keeping the boundary conditions and size of the condensed-matter region constant. By integration it is given as $D_a = G_a - \sum_b (g_{ab} - \delta_{ab})N_b$, where G_a is interpreted as the number of available single particle states of species a when no particle is presented in the system.

Wu [6] used the above dimension of the Hilbert space in order to consider the grand partition function Q of an ideal gas with FES, defined as

$$Q = \sum_{N=0}^{\infty} z^N Q_N \,, \tag{1}$$

where z is the fugacity defined by $z \equiv \exp(\beta\mu)$ with $\beta \equiv 1/kT$ and μ being the chemical potential, and N the total number of particles defined by $\mu N \equiv \sum_a \mu_a N_a$. And the canonical partition function Q_N is defined as

$$Q_N = \sum_{\{N_a\}} W(\{N_a\}) e^{-\beta E(\{N_a\})}, \tag{2}$$

$$W(\{N_a\}) = \prod_a \frac{[D_a + N_a - 1]!}{N_a![D_a - 1]!} \equiv e^{S/k}, \tag{3}$$

$$E(\{N_a\}) = \sum_a \varepsilon_a N_a \,. \tag{4}$$

Substituting Eqs. (2)–(4) into Eq. (1), Wu first considered the most probable distribution of Q, taking the extremum condition $\frac{\delta}{\delta N_a}[\ln W(\{N_a\}) + \sum_a \beta(\varepsilon_a - \mu_a)N_a] = 0$, which yields the famous Wu distribution function w_a:

$$w_a = 1/n_a - \sum_b g'_{a,b} n_b/n_a \,, \tag{5}$$

$$(1 + w_a)\prod_b \left(\frac{w_b}{1 + w_b}\right)^{g_{a,b}} = e^{\beta(\varepsilon_a - \mu_a)}, \tag{6}$$

where $n_a \equiv N_a/G_a$ and $g'_{a,b} \equiv g_{a,b}G_b/G_a$. For an identical FES-particle system, taking $g_{a,b} = g\delta_{a,b}$ and $\mu_a = \mu$, Eqs. (5) and (6) turn out to be

$$n_a = \frac{1}{w_a + g}, \qquad w_a^g(1 + w_a)^{1-g} = e^{\beta(\varepsilon_a - \mu)}. \tag{7}$$

Substituting Eqs. (5) and (6) into the thermodynamic potential Ω and the total number N, respectively, one obtains

$$\Omega = -PV = -kT\sum_a G_a \ln\left(\frac{1 + w_a}{w_a}\right), \tag{8}$$

$$N = \sum_a G_a \frac{1}{w_a + g} \,, \tag{9}$$

which is valid for all cases with different species.

Now let us convert the summation over states in Eq. (2) into that over momentum. To do this I convert the summation over states in Q_N into that over momentum as

$$Q_N = \sum_{\{n_{\mathbf{p}}\}, N = \sum_{\mathbf{p}} n_{\mathbf{p}}} g(\{n_{\mathbf{p}}\}) e^{-\beta E(\{n_{\mathbf{p}}\})}, \tag{10}$$

where $g(\{n_{\mathbf{p}}\})$ is the number of states corresponding to $\{n_{\mathbf{p}}\}$ [20]. Substitution of Eq. (10) into Eq. (1) yields

$$Q = \sum_{N=0}^{\infty} \sum_{\{n_{\mathbf{p}}\}, N=\sum_{\mathbf{p}} n_{\mathbf{p}}} g(\{n_{\mathbf{p}}\}) z^N e^{-\beta E(\{n_{\mathbf{p}}\})}. \quad (11)$$

When one considers the Q for Fermi and Bose gases, it is easy to perform since $g(\{n_{\mathbf{p}}\}) = 1$. In these cases, the grand partition functions can be factorized as $Q = \prod_{\mathbf{p}} Q_{\mathbf{p}}$, where $Q_{\mathbf{p}} = 1 + ze^{-\beta\varepsilon_{\mathbf{p}}}[= (1 - ze^{-\beta\varepsilon_{\mathbf{p}}})^{-1}]$ for fermion (boson) [20]. However, even if we impose the same relation $g(\{n_{\mathbf{p}}\}) = 1$ for an ideal gas with FES, the procedure is not that simple since one must have another constraint coming from the constraint for D_a. Thus, the problem becomes much harder. But, if possible, what is $Q_{\mathbf{p}}$ for an ideal gas with FES? To answer this, instead of using the above procedure, I follow Wu's argument [6].

To convert the summation over states in Eqs. (8) and (9) into that over momentum, suppose the factorized form of the grand partition function at the most probable distribution. From this, I can define the average occupation number $\langle n_{\mathbf{p}} \rangle$ as

$$\langle n_{\mathbf{p}} \rangle = -\frac{\partial}{\partial(\beta\varepsilon_{\mathbf{p}})} \ln Q = -\frac{\partial}{\partial(\beta\varepsilon_{\mathbf{p}})} \ln Q_{\mathbf{p}}. \quad (12)$$

Defining as $\langle n_{\mathbf{p}} \rangle \equiv 1/(W_{\mathbf{p}} + g)$ at the most probable distribution and substituting this into Eq. (12), I can integrate it with respect to $\beta\varepsilon_{\mathbf{p}}$. For an identical FES-particle case, using the relation

$$W_{\mathbf{p}}^g (1 + W_{\mathbf{p}})^{1-g} = e^{\beta(\varepsilon_{\mathbf{p}} - \mu)}, \quad (13)$$

I end up with $Q_{\mathbf{p}} = (1 + W_{\mathbf{p}})/W_{\mathbf{p}}$, where we have used simple relations: $\beta(\varepsilon_{\mathbf{p}} - \mu) = g \ln W_{\mathbf{p}} + (1 - g)\ln(1 + W_{\mathbf{p}})$ and $d(\beta\varepsilon_{\mathbf{p}}) = \frac{W_{\mathbf{p}} + g}{W_{\mathbf{p}}(1 + W_{\mathbf{p}})} dW_{\mathbf{p}}$. Hence I obtain the thermodynamic potential Ω and the total number N in the momentum representation:

$$\Omega = -PV = -kT \sum_{\mathbf{p}} \ln\left(\frac{1 + W_{\mathbf{p}}}{W_{\mathbf{p}}}\right), \quad (14)$$

$$N = \sum_{\mathbf{p}} \frac{1}{W_{\mathbf{p}} + g}. \quad (15)$$

This result holds valid for more general cases with mutual statistics $g_{\mathbf{p},\mathbf{q}}$ as well, where one has

$$(1 + W_{\mathbf{p}}) \prod_{\mathbf{q}} \left(\frac{W_{\mathbf{q}}}{1 + W_{\mathbf{q}}}\right)^{g_{\mathbf{p},\mathbf{q}}} = e^{\beta(\varepsilon_{\mathbf{p}} - \mu)}. \quad (16)$$

This was first conjectured by Bernard and Wu as a possible generalization of the CSM [7].

I now note the following theorems:

Theorem 1: Denote the pressure by $P/kT = (1/V) \times \sum_{\mathbf{p}} \ln((1 + W_{\mathbf{p}})/W_{\mathbf{p}}) \equiv F(z)$. Then $N/V = (1/V) \times \sum_{\mathbf{p}} \langle n_{\mathbf{p}} \rangle = z\frac{\partial}{\partial z}F(z)$.

Theorem 2: The second order fluctuation $\langle(\Delta n_{\mathbf{p}})^2\rangle \equiv \langle n_{\mathbf{p}}^2 \rangle - \langle n_{\mathbf{p}} \rangle^2 = -\frac{\partial}{\partial(\beta\varepsilon_{\mathbf{p}})}\langle n_{\mathbf{p}} \rangle$ is given by $\langle(\Delta n_{\mathbf{p}})^2\rangle = \langle n_{\mathbf{p}} \rangle (1 - g\langle n_{\mathbf{p}} \rangle)[1 + (1 - g)\langle n_{\mathbf{p}} \rangle]$ for an identical FES-particle gas.

The first theorem is well known and a classic result in QSM [20]. However, to show that it indeed holds valid for ideal FES gases is not so trivial since there is the nontrivial Wu distribution. The second theorem interpolates between classic results of fermion and boson cases [20]. This type of relation was first noted by Rajagopal [11]. The proof of these theorems is straightforward using Eq. (13). So, I omit them here.

Let us calculate Eqs. (14) and (15). Since the summation is converted to integration as $\sum_{\mathbf{p}} = \frac{V}{(2\pi\hbar)^D} \int d^D p = \frac{V}{(2\pi\hbar)^D} S_D \int p^{D-1} dp$, where $S_D \equiv 2\pi^{D/2}/\Gamma(D/2)$, if one assumes free FES particles as $\varepsilon_{\mathbf{p}} = \mathbf{p}^2/2m$, then the density of states (DOS) is given by $N_D(\varepsilon) = (\frac{m}{2\pi\hbar^2})^{D/2} \times \frac{1}{\Gamma(D/2)} \varepsilon^{(D-2)/2}$ and $\frac{1}{V}\sum_{\mathbf{p}} = \int d\varepsilon N_D(\varepsilon)$. Equations (14) and (15) are represented with respect to energy ε as

$$\frac{P}{kT} = \int_0^{\infty} d\varepsilon N_D(\varepsilon) \ln\left(\frac{1 + W(\varepsilon)}{W(\varepsilon)}\right), \quad (17)$$

$$\frac{N}{V} = \int_0^{\infty} d\varepsilon N_D(\varepsilon) \frac{1}{W(\varepsilon) + g}. \quad (18)$$

By changing variables, they are transformed again to the following:

$$\frac{P}{kT} = \int_{W_0}^{\infty} dW N_D[\varepsilon(W)] \frac{W + g}{W(1 + W)} \ln\left(\frac{1 + W}{W}\right), \quad (19)$$

$$\frac{N}{V} = \int_{W_0}^{\infty} dW N_D[\varepsilon(W)] \frac{1}{W(1 + W)}, \quad (20)$$

where $W_0^g (1 + W_0)^{1-g} = z^{-1}$. The last type of expressions was first obtained by Murthy and Shanker for the CSM [10].

Now I find the following remarkable fact:

Theorem 3: In a D-dimensional identical FES-particle gas, we have $Q = Q_F^g Q_B^{1-g}$. Therefore $P = gP_F + (1 - g)P_B$, and $N = N_F + N_B$ with $\mu = g\mu_F + (1 - g)\mu_B$.

To prove this theorem I follow the argument of Murthy and Shanker [10]. Define chemical potential μ as $\mu \equiv g\mu_F + (1 - g)\mu_B$ such that $W_0 \equiv e^{-\beta\mu_F}$ and $1 + W_0 \equiv e^{-\beta\mu_B}$. Define $W(\varepsilon) \equiv e^{\beta(\varepsilon - \mu_F)}$ and $1 + W(\varepsilon) \equiv e^{\beta(\varepsilon - \mu_B)}$ such that the relation $W(\varepsilon)^g[1 + W(\varepsilon)]^{1-g} = e^{\beta(\varepsilon - \mu)}$ is automatically satisfied. Next, I find $\frac{P}{kT} = \int_0^{\infty} d\varepsilon N_D(\varepsilon) \ln(\frac{1 + W(\varepsilon)}{W(\varepsilon)}) = g\int_0^{\infty} d\varepsilon N_D(\varepsilon) \ln(\frac{1 + W(\varepsilon)}{W(\varepsilon)}) + (1 - g)\int_0^{\infty} d\varepsilon N_D(\varepsilon) \ln(\frac{1 + W(\varepsilon)}{W(\varepsilon)}) = g\int_0^{\infty} d\varepsilon N_D(\varepsilon) \times \ln(1 + 1/W(\varepsilon)) - (1 - g)\int_0^{\infty} d\varepsilon N_D(\varepsilon) \ln(1 - [1 + W(\varepsilon)]^{-1})$. Substituting $1 + W^{-1} = 1 + e^{-\beta(\varepsilon - \mu_F)}$ and $1 - (1 + W)^{-1} = 1 - e^{-\beta(\varepsilon - \mu_B)}$ into the above provides the theorem. Hence, $P = gP_F + (1 - g)P_B$, which holds true for any dimension. Using properties of the logarithm to this, $Q = Q_F^g Q_B^{1-g}$ is easily followed. Since $\frac{\partial}{\partial\mu} = \frac{1}{g}\frac{\partial}{\partial\mu_F} + \frac{1}{1-g}\frac{\partial}{\partial\mu_B}$, I find $\frac{\partial P}{\partial\mu} = \frac{\partial}{\partial\mu}[gP_F + (1 - g)P_B] = \frac{\partial P_F}{\partial\mu_F} + \frac{\partial P_B}{\partial\mu_B}$. Hence, $N = N_F + N_B$.

This theorem was also first mentioned by Murthy and Shanker within the CSM in one dimension [see Eq. (25) in Ref. [10]]. However, they had to use the basic properties of the CSM to reach the expressions and to prove the theorem. On the other hand, I do not use any model dependent properties in the above proof, apart from the free identical FES-particle gas assumption. Hence, it is very general. For example, if $D = 2$ in our problem, then using the constant DOS, Eqs. (19) and (20) become exactly the equivalent expressions of Murthy and Shanker [see Eq. (27) in Ref. [10]]. Thus, I conclude that in this sense the CSM falls into the same universality class of our theorem as a special case.

Let us consider the fugacity expansions of the pressures. From the general formula Eq. (17) and the above theorem, one can get the following expressions of P_F and P_B:

$$\frac{P_F}{kT} = \int_0^\infty d\varepsilon N_D(\varepsilon) \ln(1 + z_F e^{-\beta\varepsilon}), \tag{21}$$

$$\frac{P_B}{kT} = -\int_0^\infty d\varepsilon N_D(\varepsilon) \ln(1 - z_B e^{-\beta\varepsilon}). \tag{22}$$

As is well known [20], the above equations can be expanded with respect to fugacities z_F and z_B, respectively, as the following:

Theorem 4:

$$\frac{P_F}{kT} = \frac{1}{\lambda^D} f_{D/2+1}(z_F) \quad \text{and} \quad \frac{P_B}{kT} = \frac{1}{\lambda^D} g_{D/2+1}(z_B), \tag{23}$$

where λ is the thermal length defined by $\lambda \equiv \sqrt{2\pi\hbar^2/mkT}$, and the f and g functions are defined by

$$f_{D/2+1}(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1} z^l}{l^{D/2+1}}$$
$$\text{and} \quad g_{D/2+1}(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^{D/2+1}}. \tag{24}$$

Let us consider whether or not condensation occurs in an ideal FES gas. This is answered by considering singularities in the functions. As is well known, the g functions have a singular part when $D > 2$ [20]. For example, if $D = 3$, then the singular part comes from $\varepsilon = 0$ and is stemmed out as $-\frac{1}{V}\ln(1 - z_B)$ such that $P_B/kT = g_{D/2+1}(z_B) - \frac{1}{V}\ln(1 - z_B)$. So, as $z_B \to 1$, this part becomes macroscopically significant. In our case, since $z_B^{-1} = 1 + W_0$, this condition is equivalent to the condition, $W_0 \to 0$. Thus, zeros of W_0 provide the singularities of the bosonic pressure P_B. On the other hand, there exist no such singularities in the fermionic pressure P_F.

Let us find such singularities in the pressure P. As is shown below, it is exactly equivalent to those found by Sutherland in the pressure for the CSM in his remarkable paper [19]. If P is regarded as a complex function of z, singularities occur as branch cuts in the z plane. To find such branch cuts, consider Eq. (13) by taking $\varepsilon = 0$, which gives $W_0^g(1 + W_0)^{1-g} = z^{-1}$. Differentiating it with respect to z yields $-1 = \frac{W_0+g}{W_0(1+W_0)} \frac{\partial W_0}{\partial \ln z}$. So, a nontrivial singularity occurs at $W_0 = -g$, which corresponds to a singularity in density since the denominator of Eq. (18) vanishes. At this point, branch cuts in z are given from $z_{0,\pm} = e^{\pm i\pi g}/\{g^g(1 - g)^{1-g}\}$ to infinity, which are exactly equivalent to what Sutherland found (see Fig. 1 on page 253 of his paper dated 1971 in Ref. [19]). From this I find $z_{B,0} = 1/(1 - g) \geq 1$ and $z_{F,0} = -1/g \leq -1$ since $z_0 = z_{F,0}^g z_{B,0}^{1-g}$ and $0 \leq g \leq 1$. Thus, the singularity of P_B lies at $-\frac{1}{V}\ln(1 - z_B) = -\frac{1}{V}\ln[-g/(1 - g)] = \pm\frac{1}{V} i\pi - \frac{1}{V}\ln[g/(1 - g)]$. Therefore, there is no singularity on the positive real z axis except $g = 0$ in the thermodynamic limit of $V \to \infty$. Hence, I have the following theorem:

Theorem 5: There is no condensation of identical FES-particle gases of $0 < g \leq 1$, except $g = 0$.

For example, consider an identical semion gas of $g = 1/2$. Solving $W_0^{1/2}(1 + W_0)^{1/2} = 1/z$ for W_0, I get $W_0 + \frac{1}{2} = \sqrt{1/z^2 + 1/4}$. If $W_0 = -1/2$, then there are two branch cuts extending from $z = \pm 2i$ to infinity. So, there are no singularities on the positive real z axis. Hence there is no condensation of identical semion gas.

In conclusion, I have discussed QSM of the FES gases in higher dimensions, generalizing the Wu distribution to give thermodynamic potential and density of the system in the momentum representation. I have found several basic theorems for such gases as theorems 1–5. Theorem 3 especially characterizes the universality class of the ideal FES gases in higher dimensions, and theorem 5 states that no condensation phenomena exist in $D = 1, 2$ and in $D = 3$, except the $g = 0$ boson case. This result is not surprising since the Wu distribution function holds a fermionic character for $0 < g \leq 1$ such that $\langle n_{\mathbf{p}} \rangle = 1/g$ $(= 0)$ for $\varepsilon \leq \mu$ $(\varepsilon > \mu)$ at very low temperature. Therefore, I finally conjecture that condensation occurs only when one considers nonidentical FES-particle systems with two and three components such as semion-antisemion gases and 1/3-1/3-(-2/3) triplet gases as generalizations of Cooper pairs, quarks, and Kosterlitz-Thouless transitions. It is very interesting to understand what type of interaction produces the properties of FES gases in higher dimensions.

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[1] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).
[2] A. Dasnierés de Veigy and S. Ouvry, Phys. Rev. Lett. **72**, 600 (1994).

[3] M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. **72**, 3629 (1994).
[4] M. D. Johnson and G. S. Canright, Phys. Rev. B **49**, 2947 (1994).
[5] D. Li and S. Ouvry, Nucl. Phys. **B430**, 564 (1994).
[6] Y.-S. Wu, Phys. Rev. Lett. **73**, 922 (1994).
[7] D. Bernard and Y.-S. Wu, in *New Developments of Integrable Systems and Long-Ranged Interaction Models,* edited by M.-L. Ge and Y.-S. Wu (World Scientific, Singapore, 1995) , p. 10; Y.-S. Wu, ibid., p. 159.
[8] S. B. Isakov, Phys. Rev. Lett. **73**, 2150 (1994).
[9] C. Nayak and F. Wilczek, Phys. Rev. Lett. **73**, 2740 (1994).
[10] M. V. N. Murthy and R. Shankar, Phys. Rev. Lett. **73**, 3331 (1994).
[11] A. K. Rajagopal, Phys. Rev. Lett. **74**, 1048 (1995).
[12] D. Sen and R. K. Bhaduri, Phys. Rev. Lett. **74**, 3912 (1995).
[13] Y.-S. Wu and Y. Yu, Phys. Rev. Lett. **75**, 890 (1995).
[14] T. Fukui and N. Kawakami, Phys. Rev. B **51**, 5239 (1995).
[15] Y. Kato and Y. Kuramoto, J. Phys. Soc. Jpn. **65**, 77 (1996).
[16] A. Fayyazuddin and D. Li, Phys. Rev. Lett. **76**, 1707 (1996).
[17] R. E. Prange and S. M. Girvin, *The Quantum Hall Effect* (Springer-Verlag, New York, 1987); F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[18] F. Calogero, J. Math. Phys. **10**, 2191 (1969); **10**, 2197 (1969).
[19] B. Sutherland, J. Math. Phys. **12**, 247 (1971); **12**, 250 (1971); Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972).
[20] K. Huang, *Statisitcal Mechanics* (Wiley, New York, 1987), 2nd ed.; L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1986), Part 1, 3rd ed.