TIGHT-BINDING MODEL FOR THE QUANTUM LADDER IN A MAGNETIC FIELD

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A tight-binding model is formulated for the calculation of the electronic structure and the ground state energy of the quantum ladder under a magnetic field, where the magnetic flux at the nth plaquette is given by $\phi_n$. First, the theory is applied to obtain the electronic spectra of the quantum ladder models with particular magnetic fluxes such as uniform magnetic fluxes, $\phi_n = 0$ and 1/2, and the staggered magnetic flux, $\phi_n = (-1)^{n+1}\phi_0$. From these, it is found that as the effect of electron hopping between two chains — the anisotropy parameter $r = t_{xy}/t_x$ — is increased, there are a metal-semimetal transition at $r = 0$ and a semimetal-semiconductor transition at $r = 2$ in the first case, and metal-semiconductor transitions at $r = 0, 1, 2$ in the second and third cases. These transitions are thought of as a new category of metal-insulator transition due to the hopping anisotropy of the system. Second, using the spectrum, the ground state energy is calculated in terms of the parameter $r$. It is found that the ground state energy in the first case diverges as $r$ becomes arbitrarily large, while that in the second and third cases can have the single or double well structure with respect to $r$, where the system is stable at some critical value of $r = r_c$ and the transition between the single and double well structures is associated with whether $t_c$ is less than a critical value of $t_{cr}$. The latter cases are very reminiscent of physics in polyacetylene studied by Su, Schrieffer and Heeger.

1. Introduction

There has been a current interest on the electron transport in quasi-one-dimensional mesoscopic systems such as the ladder structure — the quantum ladder model\(^1-4\) (Fig. 1). The main issue here is to investigate whether or not there is electron localization due to the presence of a random magnetic field $B_n$, where the associated magnetic flux $\phi_n$ at the $n$th plaquette is given by $\phi_n = eB_nS_n/\hbar c$ with $S_n$ being the area of the plaquette.

In the quasi-one-dimensional conductor under a random magnetic field, it has been conjectured that all the eigenstates are localized and therefore there is no mobility edge in the spectrum.\(^1-3\) To investigate this problem Avishai and Luck\(^1-3\) formulated a Kronig-Penny type model for the quantum ladder where the system is constructed by a ladder-shaped network of quantum wires. However, there has

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been no investigation on the tight-binding analog of this model. The reason seems that there has been no good method in obtaining the electronic spectrum of the quantum ladder system under periodic and nonperiodic magnetic fluxes. And in spite of the simple appealing of the model it seems that it has not been studied in detail even for the quantum ladder with periodic magnetic fluxes. Thus, it seems to be worth investigating the tight-binding analog of the quantum ladder under periodic magnetic fields, since we do not know much about the characters of the spectrum of this model.

The purpose of this paper is the following: (a) First, we will formulate the transfer matrix method for such a tight-binding model for the quantum ladder under a magnetic field, which has recently been introduced by Ohtsuki, Slevin and Ono. This method itself is not original but we believe that the scheme using the symplectic property of the $4 \times 4$ transfer matrix is original (see Sec. 3). (b) Second, to demonstrate how the theory works, we will obtain the electronic spectra for the quantum ladders with particular magnetic fluxes such as uniform magnetic fluxes, $\phi_n = 0$ and $1/2$, and the staggered magnetic flux, $\phi_n = (-1)^{n+1} \phi_0$. Although the results of the former two cases are not essentially new to provide unknown characters of the spectrum except the way of obtaining the spectrum, but we believe that that of the last case is original to give such. (c) Third, we will show that there are possibly various kinds of metal–insulator (M–I) transitions due to the electron hopping anisotropy of the system. We would like to emphasize that it may be thought of as a new category of M–I transition different from the known M–I transitions due to Peierls transition, Anderson localization, Mott transition, and the topological localization. (d) Fourth, we will also calculate the ground state energy of the three systems, respectively.

The organization of the present paper is the following. In Sec. 2 the tight-binding model for the quantum ladder is introduced. In Sec. 3 the transfer matrix method is formulated for the quantum ladders where the transfer matrices are given by $4 \times 4$ matrices associated with the double chain systems. And the scheme for...
obtaining the spectrum for electrons in the quantum ladders is constructed, using the symplectic property of the transfer matrices. In Secs. 4–6 the method is applied to obtain the electronic structures and the ground state energies for the quantum ladders with uniform magnetic fluxes of \( \phi_n = 0 \) and \( 1/2 \), and with the staggered magnetic flux, \( \phi_n = (-1)^{n+1} \phi_0 \), respectively. These are just demonstrations for the proposed mathematical method. In Sec. 7 a summary will be made.

2. Model Hamiltonian for the Quantum Ladder

2.1. Quantum ladder model

Let us first model our system of the quantum ladder (i.e. the quantum ladder model).\(^1\)\(^-\)\(^4\) Let us define a ladder structure such that there are two parallel chains connected by \( N_e \) bridges between the chains, which are numbered from 1 to \( N_e \) (Fig. 1). Therefore, there are \( 2N_e \) vertices in the ladder. Let us put, for the sake of simplicity, only one orbital at each vertex site so that there are totally \( 2N_e \) orbitals in our model of the quantum ladder. Let us denote by \( \psi_n \) (\( \phi_n \)) the orbital at site \( n \) in the single chain A(B). By superposition of the orbitals, the wave function is given by

\[
|\Psi\rangle = \sum_{n,\sigma=1,1} \psi_n a_{n\sigma}^\dagger |0\rangle + \phi_n b_{n\sigma}^\dagger |0\rangle ,
\]  

(2.1)

where \( a_{n\sigma}^\dagger (b_{n\sigma}^\dagger) \) means the electron creation operator with spin \( \sigma \) at site \( n \) in chain A (B), which obeys the usual anticommutation relation.

2.2. Hamiltonian for electrons in the quantum ladder under a magnetic field

Let us define the Hamiltonian \( H \) of the system. We allow electrons to hop through bonds or nearest neighbor sites of the quantum ladder. To adjust with the quantum ladder model under a magnetic field, we take a gauge to put a phase factor \( e^{i\theta_n} (e^{-i\theta_n}) \) for the electron hopping from chain A (B) to chain B (A) at site \( n \) (Fig. 1). And we assume that the magnetic flux in each plaquette is given by \( \phi_n = eB_nS_n/hc \) where \( B_n \) is the magnetic field piercing the \( n \)th plaquette with area \( S_n \) (Fig. 2). Hence, we have the relation between the magnetic flux at the \( n \)th plaquette and the phase:

\[
\phi_n = \frac{1}{2\pi} (\theta_{n+1} - \theta_n) ,
\]  

(2.2)

This gives the Hamiltonian \( H_e \) for electrons in the system as

\[
H_e = \sum_{n,\sigma=1,1} \left\{ -t_x (a_{n+1\sigma}^\dagger a_{n\sigma} + b_{n+1\sigma}^\dagger b_{n\sigma}) - t_y (e^{i\theta_n} b_{n\sigma}^\dagger a_{n\sigma} + \text{h.c.}) \right\} ,
\]  

(2.3)

where \( t_x (t_y) \) means the electron hopping potential in (between) the chains.\(^4\)

Let us now introduce the magnetic energy \( H_m \) to the system. The knowledge of Abelian gauge field theory\(^9\)\(^-\)\(^11\) tells us that the magnetic energy at a plaquette
between the \((n+1)\)th and the \(n\)th bridges is given by the electron hopping around the plaquette, which is 
\[-t_y^2 t_y^2 e^{i(\theta_n+1-\theta_n)} + \text{c.c.} = -2t_y^2 t_y^2 \cos(\theta_{n+1} - \theta_n)\].
Thus, it is given as the sum of the total hopping around each plaquette over the double chain system (Fig. 2):

\[
H_m = \sum_n \left\{ \frac{A}{2} \left( \frac{\partial \theta_n}{\partial \tau} \right)^2 + t_y^2 t_y^2 \cos(\theta_{n+1} - \theta_n) \right\},
\]

where \(A\) is a constant and the energy origin is shifted so that it turns to be zero when no magnetic field. In this way the total Hamiltonian of the system is given by

\[
H = H_e + H_m
= \sum_{n,\sigma=\uparrow,\downarrow} \left\{ -t_x (a_{n+1\sigma}^\dagger a_{n\sigma} + b_{n+1\sigma}^\dagger b_{n\sigma}) - t_y (e^{i\theta_n} b_{n\sigma}^\dagger a_{n\sigma} + \text{h.c.}) \right\}
+ \sum_n \left\{ \frac{A}{2} \left( \frac{\partial \theta_n}{\partial \tau} \right)^2 + t_y^2 t_y^2 \left[ 1 - \cos(\theta_{n+1} - \theta_n) \right] \right\},
\]

\[\text{(2.5)}\]

2.3. **Hamiltonian for electrons in the planar spin chain**

We are now going to show that the above Hamiltonian is equivalent to that for electrons in a planar spin chain. If we change the basis of the wave functions to that in the spinor representation:

\[
\Psi_{n\sigma} = \begin{pmatrix} a_{n+1\sigma} \\ a_{n\sigma} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} a_{n\sigma} \\ b_{n\sigma} \end{pmatrix},
\]

the above \(H_e\) in the quantum ladder model turns out to be equivalent to the Hamiltonian for electrons in a spin chain1:

\[
H_e = \sum_{n,\sigma=\uparrow,\downarrow} \left\{ -t_x (\Psi_{n+1\sigma}^\dagger \Psi_{n\sigma} + \text{h.c.}) - t_y \Psi_{n\sigma}^\dagger S_n \cdot \sigma \Psi_{n\sigma} \right\},
\]

\[\text{(2.7)}\]
where $\sigma = (\sigma_x, \sigma_z)$ with $\sigma_x$ and $\sigma_z$ being the Pauli spin matrices, and the planar spin $S_n$ at site $n$ in the $x$-$z$ plane is defined as

$$S_n = (S_x(n), S_z(n)) = (\sin \theta_n, \cos \theta_n).$$

(2.8)

Here we have assumed that the $x$-($z$-) direction is along (perpendicular to) the spin chain and $\theta_n$ means the rotation angle of the spin at site $n$ in the $x$-$z$ plane and is measured from the $z$ direction (Fig. 3).

![Fig. 3. The planar spin chain model.](image)

We now find that the above magnetic energy becomes equivalent to the interaction energy of the planar spins:

$$H_s = -J \sum_n S_{n+1} \cdot S_n + \sum_n \frac{A}{2} \left( \frac{\partial S_n}{\partial t} \right)^2,$$

(2.9)

where $J = t_x^2/t_y^2$. Thus, in the spinor representation the Hamiltonian is given by

$$H = H_e + H_s$$

$$= \sum_{n, \sigma=1,\downarrow} \left\{ -t_x (\Psi_{n+1\sigma}^\dagger \Psi_{n\sigma} + h.c) - t_y \Psi_{n\sigma}^\dagger S_n \cdot \sigma \Psi_{n\sigma} \right\}$$

$$- J \sum_{n} S_{n+1} \cdot S_n + \sum_{n} \frac{A}{2} \left( \frac{\partial S_n}{\partial t} \right)^2.$$

(2.10)

To show more clearly the relationship between the quantum ladder under a magnetic field and the planar spin chain, we would like to give some examples that will be investigated in later sections. For example, (a) if $\theta_n = \theta_0 = \text{constant}$, then $\phi_n = 0$ from Eq. (2.2), which means no magnetic flux case [Fig. 4(a)]. This corresponds to a planar spin chain where all spins are aligned to the same direction [Fig. 4(b)] since we have

$$S_n = (\sin \theta_0, \cos \theta_0).$$

(2.11)
(b) If \( \theta_n = 2\pi \phi n \) with \( \phi \) being constant, then \( \phi_n = \phi \) from Eq. (2.2), which means a uniform magnetic flux case [Fig. 5(a)]. This corresponds to the planar spin chain where the spins are rotated with frequency \( \phi \) in the \( x-z \) plane [Fig. 5(b)] since we have

\[
S_n = (\sin(2\pi \phi n), \cos(2\pi \phi n)).
\]  

(2.12)
(c) If $\theta_n = (-1)^n \theta_0$, then $\phi_n = (-1)^{n+1} \phi_0 = (-1)^{n+1} \theta_0 / \pi$ from Eq. (2.2), which means a staggered magnetic flux case [Fig. 6(a)]. This corresponds to a tiltedly aligned planar spin chain [Fig. 6(b)] since we have

$$S_n = ((-1)^n \sin \theta_0, \cos \theta_0) \quad (2.13)$$

![Diagram](a)

![Diagram](b)

Fig. 6. (a) The quantum ladder model without a staggered magnetic flux. (b) The tilted planar spin chain.

(d) And if $\theta_n$ is a random number, then so is $\phi_n$ [Fig. 7(a)]. This corresponds to a planar spin chain where all spins are aligned randomly [Fig. 7(b)]. The above three cases of (a)–(c) will be studied in Secs. 4–7, respectively.

![Diagram](a)

![Diagram](b)

Fig. 7. (a) The quantum ladder model without a random magnetic flux. (b) The randomly oriented planar spin chain.
3. Transfer Matrix Method for the Quantum Ladder

3.1. Transfer matrix

Let us formulate the transfer matrix method for the calculation of the electronic spectrum of the system. Applying the Hamiltonian $H_e$ [Eq. (2.3) or Eq. (2.7)] to the Schrödinger equation, $H_e|\Psi\rangle = E|\Psi\rangle$, we obtain a pair of eigenvalue equations:

$$-t_x(\psi_{n+1} + \psi_{n-1}) - t_y e^{i\theta_n} \varphi_n = E\psi_n, \quad -t_x(\varphi_{n+1} + \varphi_{n-1}) - t_y e^{-i\theta_n} \psi_n = E\varphi_n. \quad (3.1)$$

Similarly, in the spinor representation, Eq. (2.7) turns out to be

$$-t_x(\Psi_{n+1} + \Psi_{n-1}) - t_y (\cos \theta_n \sigma_z + \sin \theta_n \sigma_x) \Psi_n = E\Psi_n, \quad (3.2)$$

$$\Psi_n \equiv \begin{pmatrix} \psi_{+,n} \\ \psi_{-,n} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \psi_n \\ \varphi_n \end{pmatrix}. \quad (3.3)$$

Let us now define a four-dimensional column vector $\Psi_n$ at site $n$ as $\Psi_n \equiv \{\psi_n, \psi_{n-1}, \varphi_n, \varphi_{n-1}\}$. Then, using Eq. (2.3) together with trivial relations, $\psi_n = \psi_n$ and $\varphi_n = \varphi_n$, we get

$$\Psi_{n+1} = M_n \Psi_n, \quad M_n = \begin{pmatrix} A_n & U_n \\ U_n^\dagger & A_n \end{pmatrix}. \quad (3.4)$$

Here $M_n$ is the $4 \times 4$ transfer matrix with the $2 \times 2$ matrices:

$$A_n = \begin{pmatrix} -E/t_x & -1 \\ 1 & 0 \end{pmatrix}, \quad U_n = \begin{pmatrix} V_n & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.5)$$

where $V_n \equiv r e^{i\theta_n}$ with the anisotropy parameter $r \equiv t_y/t_x$. And similarly in the spinor representation, let us define the four-dimensional column vector $\Psi_n$ at site $n$ as $\Psi_n \equiv \{\psi_{+,n}, \psi_{+,n-1}, \psi_{-,n}, \psi_{-,n-1}\}$. We then get

$$\Psi_{n+1} = M_n \Psi_n, \quad M_n = \begin{pmatrix} A_n & U_n \\ U_n^\dagger & A_n \end{pmatrix}, \quad (3.6)$$

$$A_n (B_n) = \begin{pmatrix} -E/t_x - (+)r \cos \theta_n & -1 \\ 1 & 0 \end{pmatrix}, \quad U_n = \begin{pmatrix} -r \sin \theta_n & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.7)$$

3.2. Eigenvalues of the transfer matrix

According to the sequence of $N_e$ segments, we have to take a product of the $N_e$ transfer matrices $M$ such that $M \equiv M_{N_e} M_{N_e-1} \cdots M_1$, which is also a $4 \times 4$ matrix. If we impose the quantum ladder system to be periodic (i.e. $\psi_{n+N_e} = \psi_n$ and $\varphi_{n+N_e} = \varphi_n$), then from Eq. (3.4) [or Eq. (3.6)] we have the condition $\text{Det}[M - I_4] = 0$, where $I_4$ is the $4 \times 4$ unit matrix. It provides the wavevector $k$ in the system such
that $k = 2\pi j/N_s$ for $j = -N_s/2, \ldots, N_s/2 \mid -(N_s - 1)/2, \ldots, (N_s - 1)/2$ when $N_s$ is even (odd).

Suppose next that the system is arbitrarily large (i.e. $N_s \to \infty$). Let us impose the commensurability condition for the phase:

$$\theta_{n+q} = \theta_n. \quad (3.8)$$

It then provides a periodic structure of the magnetic flux:

$$\phi_{n+q} = \phi_n, \quad (3.9)$$

the special case of which is the uniform and rational magnetic flux case:

$$\phi_n = \phi = \frac{p}{q}, \quad \theta_n = 2\pi n \phi. \quad (3.10)$$

Thus, we have the unit cell with length $q$ such that the Bloch theorem is adapted to the system:

$$\psi_{n+q} = \rho \psi_n, \quad \varphi_{n+q} = \rho \varphi_n \quad (3.11)$$

with $\rho = \exp(ikq)$.

Applying it to the eigenvalue equation Eq. (3.4) [or Eq. (3.6)] for $M = M_s M_{s-1} \cdots M_1$, we find a $4 \times 4$ determinant $D(\rho)$ that is a fourth order polynomial of $\rho$:

$$D(\rho) \equiv \text{Det}[M - \rho I_4] = \rho^4 - a_1 \rho^3 + a_2 \rho^2 - a_3 \rho + a_4 = 0. \quad (3.12)$$

Here if four roots are written as $\rho_1, \rho_2, \rho_3,$ and $\rho_4$, then from the knowledge of linear algebra we have

$$a_1 = \text{Tr} M = \sum_{i=1}^{4} \rho_i, \quad a_2 = \sum_{i<j=1}^{4} \rho_i \rho_j, \quad a_3 = \sum_{i<j<k=1}^{4} \rho_i \rho_j \rho_k, \quad a_4 = \text{Det} M = \rho_1 \rho_2 \rho_3 \rho_4. \quad (3.13)$$

3.3. Symplectic property of the transfer matrix

The problem now is how to solve the biquadratic equation (3.10). At first glance, since the components of the matrix $M$ may be very complicated in general, there seems no hope of easily solving it, although it can be solved analytically in principle using the formula for roots of a biquadratic equation. However, it is not so but there is a convenient way to do that. Using a physical intuition — the time reversal symmetry, if an electron propagation with the wavevector $k$ in one direction is represented by $\rho$, then the reverse (time reversal) propagation with the wavevector $-k$ is represented by $\rho^{-1}$ as well. Therefore, the latter should be also accessible. Hence, $\rho^{-1}$ must be an eigenvalue of Eq. (3.10) such that $D(\rho^{-1}) = \rho^{-4}(a_4 \rho^4 - a_3 \rho^3 + a_2 \rho^2 - a_1 \rho + 1) = 0$. 
This situation imposes the particular condition on the matrix $M$:

$$M^\dagger J M = J,$$  \hspace{1cm} (3.14)

where $M^\dagger$ means the Hermitian conjugate of $M$, and $J = (0, 1; -1, 0)$ and $0$ is the $2 \times 2$ null matrix, respectively. This property is called the symplectic structure\cite{12} of $M$, from which we find the relation:

$$D(\rho) = \rho^4 D(\rho^{-1}),$$  \hspace{1cm} (3.15)

which provides

$$a_1 = a_3, \quad a_4 = 1.$$  \hspace{1cm} (3.16)

To prove Eq. (3.15) for the symplectic property of $M$, we usually assume $M^\dagger J M = J$, $J \equiv (0, I; -I, 0)$ with $I$ the $2 \times 2$ unit matrix. \cite{12} But in our setting of $\Psi_n$, we have to assume $J \equiv (J, 0; 0, J)$. If we define as $\Psi_n \equiv (\psi_n, \varphi_n, \psi_{n-1}, \varphi_{n-1})$, then we get the above standard one. Therefore, both definitions of $\Psi_n$ are identical to each other. We note that this property is always valid in our present models. Hence, $M$ is symplectic in our models.

### 3.4. Scheme for obtaining energy bands and density of states

By using this fact and dividing Eq. (3.12) by $\rho^2$, it is reduced to the quadratic equation:

$$x^2 - a_1 x + a_2 - 2 = 0,$$  \hspace{1cm} (3.17)

$$x = \rho + \frac{1}{\rho}.$$  \hspace{1cm} (3.18)

Therefore, its two roots are given as

$$x_{\pm} = \frac{1}{2} \left[ a_1 \pm \sqrt{a_1^2 - 4a_2 + 8} \right].$$  \hspace{1cm} (3.19)

From the relations of Eq. (3.13) and $\text{Tr}(M^2) = \sum_{i=1}^{4} a_i^2$, we find $a_1^2 - 2a_2 = \text{Tr}(M^2)$. Solving this for $a_2$, we get $a_2 = [(\text{Tr} M^2) - \text{Tr}(M^2)]/2$. Therefore, we find $D = a_1^2 - 4a_2 + 8 = 2\text{Tr}(M^2) - (\text{Tr} M)^2 + 8$. Hence, we can rewrite Eq. (3.19) in terms of $\text{Tr} M$ and $\text{Tr}(M^2)$ as

$$x_{\pm} = \frac{1}{2} \left[ \text{Tr} M \pm \sqrt{D} \right],$$  \hspace{1cm} (3.20)

$$D \equiv 2\text{Tr}(M^2) - (\text{Tr} M)^2 + 8,$$  \hspace{1cm} (3.21)

which means that there are two channels in the system denoted by $\pm$, which agrees with the Kronig-Penny model for the ladder structure.\cite{1-3} Accordingly, substituting Eq. (3.20) into Eq. (3.18) and solving for $\rho$, we are able to obtain the eigenvalues of Eq. (3.12):

$$\rho_{\pm}^{(\pm)} = \frac{1}{2} \left[ x_{\pm} \pm \sqrt{x_{\pm}^2 - 4} \right].$$  \hspace{1cm} (3.22)
where \( \rho^{(+)\pm} \) and \( \rho^{(-\pm)} \) means each two solutions for channels + and -, respectively, and \( \rho^{(+)\pm} \rho^{(-\pm)} = 1 \). So if \( |x_\pm| \leq 2 \), then \( \rho^{(\pm)} \) are imaginary lying on the unit circle, and if \( |x_\pm| > 2 \), then \( \rho^{(\pm)} \) are real such that \( \rho^{(\pm)} > 1 > \rho^{(-\pm)} > 0 \) (\( \rho^{(+)} < -1 < \rho^{(-)} < 0 \)) for \( x_\pm \) is positive (negative).

We now have a simple scheme in order to obtain the spectrum: If an energy \( E \) satisfies \( x_\pm = 2 \cos(kq) \), then the energy is allowed, otherwise it is forbidden in channel \( \pm \), respectively. This is a generalized version of the Bloch condition for the \( 2 \times 2 \) matrix \( M \): \( \text{Tr} M = 2 \cos(kq) \).

The density of states (DOS) \( D_{\pm}(E) \) is calculated using Eq. (3.20) for each channel \( \pm \), respectively:

\[
dk_\pm = \frac{1}{q} \cos^{-1} \left( \frac{x_\pm(E)}{2} \right) = -\frac{1}{q} \left( \frac{\partial x_\pm}{\partial E} \right) \sqrt{4 - x_\pm^2} dE = D_\pm(E)dE. \tag{3.23}
\]

Therefore, the total DOS is given as the sum of \( D_+(E) \) and \( D_-(E) \):

\[
D(E) = D_+(E) + D_-(E). \tag{3.24}
\]

3.5. The separated chain limit

Let us first consider the separated chain limit (i.e. \( r = 0 \)). In this limit we find that \( \text{Tr} M = 2 \text{Tr} A \) and \( \text{Tr}(M^2) = 2 \text{Tr}(A^2) = 2(\text{Tr} A)^2 - 4 \), where \( A \equiv A_qA_{q-1} \cdots A_1 \). Therefore,

\[
D = 2\text{Tr}(M^2) - (\text{Tr} M)^2 + 8 = 0. \tag{3.25}
\]

Hence, we get the trivial result:

\[
x_+ = x_- = \text{Tr} A, \tag{3.26}
\]

in consistent with the physical situation in this limit. Thus, when \( r = 0 \), the problem is nothing more than considering the two separated chains independently. Hence, it is a single chain problem using the transfer matrix method belonging to \( \text{SL}(2,R) \).

In this way the above transfer matrix method using the \( 4 \times 4 \) matrices can treat both the separated two chains and the double chain in a unified manner. This is the advantage of the present method.

4. The Quantum Ladder without Magnetic Flux

4.1. Electronic structure

Let us now consider the quantum ladder in the absence of magnetic field (i.e. \( \phi_n = 0 \)) in order to obtain its spectrum where we take \( \theta_n = 0 \) for simplicity. It is a good example to see how the theory works in the problem as the first step. This system is equivalent to the planar spin system where all spins are aligned to the z-direction [Fig. 4(b)]. From Eq. (3.4), we get

\[
\text{Tr} M = 2 \text{Tr} A \]
\[ M = M_1 = \begin{pmatrix} -E/t_x & -1 & -r & 0 \\ .51 & 0 & 0 & 0 \\ .5 - r & 0 & -E/t_x & -1 \\ .50 & 0 & 1 & 0 \end{pmatrix}. \] (4.1)

Since the $M$ is symplectic,\textsuperscript{12} we can follow the scheme to obtain the energy bands. Calculating $\text{Tr} M$ and $\text{Tr}(M^2)$ and substituting into Eqs. (3.20) and (3.21), we obtain the energy bands for the two channels,\textsuperscript{1,3}

\[ E_{\pm} = -2t_x \cos k \pm t_y, \] (4.2)

respectively. Here, $E_-(E_+)$ represents the energy band for bonding (antibonding) states between the two chains. The special case of $r = 1$ provides $E_{\pm} = -t_x(2\cos k \pm 1)$, which was previously obtained by Burdett.\textsuperscript{13} If $t_x, t_y > 0$, then $E_+ > E_-$, and if $t_x > 0, t_y < 0$, then $E_+ < E_-$. The spectrum is symmetric around $E = 0$ and if we denote by $\Delta_0$ the difference between the bottom of $E_+$ and the top of $E_-$, then it is given by

\[ \Delta_0 = E_+(0) - E_-(\pi) = 2t_x(|r| - 2). \] (4.3)

The energy bands are shown for $t_x = 1$ and $r$ ranging from 0 to 3 in Fig. 8.

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Fig. 8. Energy bands for the quantum ladder without a magnetic flux. The case of $t_x = 1$ is shown with $r$ ranging from 0 to 3. The vertical axis means the energy $E$ and the horizontal axis the wavevector $k$ in units of $\pi$. The horizontal plane means the position of the Fermi energy (i.e. $E_F = 0$).
There are three distinct characters of the system with respect to \( r \) (Fig. 8). (a) If \( r = 0 \) (the separated chain limit), then the energy bands for both channels are given by \( E_\pm = -2t_x \cos k \), respectively. Hence, the system is a metal. (b) If \( 0 < r \leq r_c \equiv 2 \), then the energy bands are separated by the difference \( 2t_y \), but they are overlapped in the region \( -|\Delta_0|/2 \leq E \leq |\Delta_0|/2 \) in the spectrum (i.e. \( \Delta_0 < 0 \)). Hence, the system is a semimetal. So, in this limit there appear the electron and hole carriers in the system. (c) And if \( r < r_c \), then both energy bands are separated such that \( E_- \leq 0 \leq E_+ \) with a band gap \( \Delta_0 \). So, the system is a semiconductor. Thus, we encounter a metal-semimetal-semiconductor transition in the quantum ladder as the effect of the electron hopping \( t_y \) between chains is increased.

We now are going to calculate the ground state energy \( E_g \) of the system. To do so, we have to first know the Fermi wavevectors. We consider only the case when the system is half filled. Now, the Fermi energy \( E_F \) lies at the band center \( E = 0 \) (i.e. \( E_F = 0 \)). Let us find out the corresponding Fermi wavevectors, \( k_F^{(\pm)} \). From Eq. (4.2), we have

\[
-2t_x \cos k_F^{(\pm)} \pm t_y = 0,
\]

which give only one solution for \( r = 0 \) since the energy bands for the two channels are degenerate, two solutions for \( 0 < r \leq 2 \) since the energy bands are semimetal like with overlapping in the region \( -|\Delta_0|/2 \leq E \leq |\Delta_0|/2 \) (Fig. 9), and no solution for \( r > 2 \) since there is a gap in the region \( -\Delta_0/2 \leq E \leq \Delta_0/2 \).

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**Fig. 9.** Positions of the Fermi wavevectors \( k_F^{(\pm)} \) for the quantum ladder without a magnetic flux.
For \(0 \leq r \leq 2\), the wavevectors are represented as

\[
\begin{align*}
    k_F^{(+)} &= \cos^{-1}\left(\frac{r}{2}\right), \\
    k_F^{(-)} &= \cos^{-1}\left(-\frac{r}{2}\right) = \pi - k_F^{(+)} = \pi - \cos^{-1}\left(\frac{r}{2}\right),
\end{align*}
\]

which reproduce \(k_F^{(+)} = \pi/2\) for \(r = 0\) and \(k_F^{(+)} = 0\) and \(k_F^{(-)} = \pi\) for \(r = 2\). These Fermi wavevectors are shown in Fig. 10.

\[
\begin{align*}
    k_F^{(+)} &= \cos^{-1}\left(\frac{r}{2}\right), \\
    k_F^{(-)} &= \cos^{-1}\left(-\frac{r}{2}\right) = \pi - k_F^{(+)} = \pi - \cos^{-1}\left(\frac{r}{2}\right),
\end{align*}
\]

4.2. The ground state energy of the system

Let us calculate the ground state energy \(E_g\) of the system. It is given by

\[
E_g = 2 \sum_{k, \text{oec}} E_-(k) + 2 \sum_{k, \text{oec}} E_+(k),
\]

where the factor 2 in front of the integrals comes from spin degeneracy. By converting the summation into the integration using \(\sum_k = (N_s/2\pi) \int dk\), Eq. (4.6) turns out to be

\[
E_g = \frac{N_s}{\pi} \int_{-k_F^{(-)}}^{k_F^{(+)}} E_-(k)dk + \frac{N_s}{\pi} \int_{-k_F^{(+)}}^{k_F^{(-)}} E_+(k)dk.
\]
Substituting Eq. (4.2) into the above we get

\[
E_g = \frac{N_s}{\pi} \int_{-k_F^{(-)}}^{k_F^{(+)}} (-2t_x \cos k - t_y) dk + \frac{N_s}{\pi} \int_{-k_F^{(-)}}^{k_F^{(+)}} (-2t_x \cos k + t_y) dk
\]

\[
= -\frac{4N_s t_x}{\pi} \left[ \int_{0}^{k_F^{(-)}} \cos kdk + \int_{0}^{k_F^{(+)}} \cos kdk \right] - \frac{2N_s t_y}{\pi} \left( k_F^{(-)} - k_F^{(+)} \right)
\]

\[
= -\frac{4N_s t_x}{\pi} \left[ \sin k_F^{(-)} + \sin k_F^{(+)} \right] - \frac{2N_s t_y}{\pi} \left( k_F^{(-)} - k_F^{(+)} \right)
\]

\[
= -\frac{8N_s t_x}{\pi} \sin k_F^{(+)} - \frac{2N_s t_y}{\pi} \left( \pi - 2k_F^{(+)} \right)
\]

\[
= -\frac{8N_s t_x}{\pi} \sqrt{1 - \left(\frac{r}{2}\right)^2} - 2N_s t_y \left[ 1 - \frac{2}{\pi} \cos^{-1} \left(\frac{r}{2}\right) \right] \equiv N_s \varepsilon_g(r), \quad (4.8)
\]

where we have used \( \sin k_F^{(-)} = \sin(\pi - k_F^{(+)}) = \sin k_F^{(+)} \) and it is valid for \( 0 \leq r \leq 2 \). Equation (4.8) gives \( \varepsilon_g(r = 0) = -8t_x/\pi \) since \( \cos^{-1}(0) = \pi/2 \), and \( \varepsilon_g(r = 2) = -2t_y = -4t_x \) since \( \cos^{-1}(1) = 0 \). And for \( r > 2 \), \( \varepsilon_g(r) = -2t_y = -2t_x r \). Thus, we have

\[
\varepsilon_g(r) / t_x = \begin{cases} 
-\frac{8}{\pi} \sqrt{1 - \left(\frac{r}{2}\right)^2} - 2r \left[ 1 - \frac{2}{\pi} \cos^{-1} \left(\frac{r}{2}\right) \right] & \text{for } 0 \leq r \leq 2 \\
-2r & \text{for } r > 2
\end{cases} \quad (4.9)
\]

This is shown in Fig. 11 for the case of \( t_z = 1 \).
We note the following: The inter-chain hopping $r$ is increased as the potential separation of the two chains is decreased. From the above result, the ground state energy, $\varepsilon_g(r)$, is a monotonously decreasing function with respect to $r$ and $\varepsilon_g(r) = \varepsilon_g(-r)$. Hence, we conclude that this system is very unstable under the introduction of the electron hopping between the two chains. Therefore, unless there is another mechanism for the two chains to repel each other, the system is collapsing into a system of the overlapping double chain with no area of the plaquette (Fig. 12). Thus, we have a kind of attractive potential between the two chains in this case.

Fig. 12. Schematic diagram of the transition to the ground state.

We did not pay any attention to many-body effects such as the electron-hole and electron-electron interactions, however. It is possible to include their effects as studied by Kohn.\textsuperscript{14} If the electron-hole interaction is taken into account, then there may appear the regime of an excitonic insulator in between the semimetal and semiconductor regimes. This is an interesting problem but out of the scope of our purpose here. So, we ignore such many-body effects in this paper.

5. The Quantum Ladder with the Uniform Magnetic Flux of $\phi_n = 1/2$

5.1. Electronic structure

Let us consider the quantum ladder with the magnetic flux of $\phi_n = 1/2$ [Fig. 13(a)]. In this case we have $\theta_n = 2\pi n \phi = \pi n$, and hence we have

$$V_n = (-1)^n r. \quad (5.1)$$

This system is equivalent to the staggered planar spin chain [Fig. 13(b)] with

$$S_n = (\sin \theta_n, \cos \theta_n) = (0, (-1)^n). \quad (5.2)$$
Fig. 13. (a) The quantum ladder model with a uniform magnetic flux of $\phi_n = 1/2$. (b) The staggered planar spin chain. The spins are aligned to the $z$-direction.

Applying it to Eq. (3.4), we have two matrices $M_1$ and $M_2$, where $M_1$ is the same as in Eq. (4.1) and $M_2$ is given by replacing $r$ by $-r$ in $M_1$. Multiplying these we get

$$M \equiv M_2 M_1 = \begin{pmatrix} E^2/t_x^2 - r^2 - 1 & E/t_x & 0 & r \\ -E/t_x & -1 & r & 0 \\ 0 & r & E^2/t_x^2 - r^2 - 1 & E/t_x \\ r & 0 & -E/t_x & -1 \end{pmatrix}.$$  \hfill (5.3)

Since $M$ satisfies the condition (3.14), we can follow the scheme to obtain the energy bands. Calculating $\text{Tr} M$ and $\text{Tr}(M^2)$, we obtain

$$\text{Tr} M = 2 \left[ \frac{E^2}{t_x^2} - r^2 - 2 \right],$$

$$\text{Tr}(M^2) = 2 \left[ \frac{E^4}{t_x^4} - 2(2 + r^2) \frac{E^2}{t_x^2} + r^4 + 4r^2 + 2 \right].$$  \hfill (5.4)

Substituting these into Eq. (3.21), we get $D = 2\text{Tr}(M^2) - (\text{Tr} M)^2 + 8 = 0$. Hence, we obtain from Eq. (3.20)

$$x_\pm = \frac{E^2}{t_x^2} - r^2 - 2,$$  \hfill (5.5)

where $x_\pm = 2 \cos 2k$ from the band condition for both channels. Solving these for $E$, we finally find

$$E^{(\pm)}_\pm(k) = \pm t_x \sqrt{r^2 + 4 \cos^2 k},$$  \hfill (5.6)
where $E^{(\pm)}_+(k)$ and $E^{(\pm)}_-(k)$ stand for the upper and lower energy bands in channels $\pm$, respectively, and hence the energy bands in both channels are degenerate with each other since $D = 0$ in this case. This means that in the presence of the magnetic flux $\phi_n = 1/2$ the two chains are decoupled to be independent with having the energy bands influenced by the magnetic field.

![Energy bands for the quantum ladder with the magnetic flux of $\phi_n = 1/2$. The case of $t_x = 1$ is shown with $r$ ranging from 0 to 2. The vertical axis means the energy $E$ and the horizontal axis the wavevector $k$ in units of $\pi$. The horizontal plane means the position of the Fermi energy (i.e. $E_F = 0$).](image)

We now observe that the spectrum is symmetric around $E = 0$ and $E_+(k) \geq E_-(k)$ with respect to $k$ unless $r = 0$ (Fig. 14). Let us denote by $\Delta_0$ the difference between the bottom of $E_+(k)$ and the top of $E_-(k)$ [i.e. $\Delta_0 = E_+(\pi/2) - E_-(\pi/2)$], which measures the magnitude of a band gap at the center of the spectrum. From Eq. (5.6), we find

$$\Delta_0 = 2t_y.$$  

(5.7)

Therefore, if the half filling (i.e. $E_F = 0$) is considered, then the system is a semiconductor since there is always a gap at $E = 0$ unless $t_y = 0$. This means that there is a metal–semiconductor transition (the M–I transition) at $t_y = 0$. In this way the presence of a magnetic flux of $\phi_n = 1/2$ stabilizes the system to be a semiconductor.
It is analogous to the Peierls stabilization in the electron–phonon system and the magnetic stabilization in the Harper model. Thus, it gives rise to a physically very different situation from the non-magnetic flux case of $\phi_n = 0$.

### 5.2. The ground state energy of the system

The magnetic energy $E_m$ of the system is given by

$$E_m = t_x^2 t_y^2 \sum_n \{1 - \cos(\theta_{n+1} - \theta_n)\} = t_x^2 t_y^2 \sum_n \{1 - \cos(2\pi \phi_n)\} = N_s t_x^2 t_y^2 \{1 - \cos(\pi)\} = 2N_s t_x^2 t_y^2.$$  

(5.8)

Let us calculate the ground state energy $E_g$ of the system. It is given by

$$E_g = 2 \sum_{k, \text{occ.}} E^{(+)}(k) + 2 \sum_{k, \text{occ.}} E^{(-)}(k) + E_m$$

$$= \frac{2N_s}{\pi} \int_{-\pi/2}^{\pi/2} E^{(+)}(k)dk + \frac{2N_s}{\pi} \int_{-\pi/2}^{\pi/2} E^{(-)}(k)dk + E_m,$$  

(5.9)

where the factor 2 in front of the integrals comes from spin degeneracy. It becomes

$$E_g = \frac{4N_s}{\pi} \int_{-\pi/2}^{\pi/2} E_-(k)dk + E_m,$$  

(5.10)

where we have used $E^{(+)}(k) = E^{(-)}(k) = E_-(k)$. Substituting Eq. (5.6) into Eq. (5.10), we obtain

$$E_g = -\frac{8N_s t_x}{\pi} \int_{-\pi/2}^{\pi/2} \sqrt{r^2 + 4\cos^2 k} dk + E_m$$

$$= \frac{8N_s}{t_x} \sqrt{r^2 + 4} \int_{0}^{\pi/2} \left(1 - \frac{4}{r^2 + 4} \sin^2 k dk + 2N_s t_y^2 = N_s \varepsilon(r) \right).$$  

(5.11)

It turns out to be

$$\varepsilon(r) = -\frac{16}{\pi m} \frac{E(m)}{\sqrt{m}} + 2t_x^2 r^2,$$  

(5.12)

where $m = 4/(r^2 + 4)$ and $E(m)$ is the complete elliptic integral of the second kind:

$$E(m) = \int_{0}^{\pi/2} \sqrt{1 - m \sin^2 k} dk.$$  

(5.13)

The ground state energy has an interesting property. Equation (5.12) is shown with respect to $r$ in Figs. 15(a)–(c) for the cases of (a) $t_x < 1$, (b) $t_x = 1$, and (c) $t_x > 1$, respectively. If some parameter region of $t_x < t_{xc} \approx 1$ is considered, then the ground state energy has the double well structure, where the minimum occurs at $r = \pm r_c$ [see Fig. 15(a)]. Hence, the ground state is twofold degenerate. On the other hand, if $t_x > t_{xc}$, then the ground state energy has the single well structure with a minimum at $r = 0$ [see Fig. 15(c)]. And there is a transition in between them [see Fig. 15(b)].
Fig. 15. The ground state energy $E_g$ of the quantum ladder with a uniform magnetic flux of $\phi_o = 1/2$. (a) $t_\sigma = 0.8$; (b) $t_\sigma = 1$; (c) $t_\sigma = 1.3$. 
Fig. 15. (Continued)

Fig. 16. Schematic diagram of the transition to the ground state. (a) For $t_x < t_{xc}$, the system is stable when the finite area of the plaquette is maintained. (b) For $t_x > t_{xc}$, the system is stable when the two chains are well separated in distance.
This situation is very reminiscent of that in polyacetylene.\textsuperscript{16-18} Indeed, it is analogous to each other except that the parameter $r$ here is the hopping between two parallel chains while the $r$ is the lattice deformation parameter in the linear chain of polyacetylene.\textsuperscript{16-18}

From this fact, we conclude the following: For the case of $t_z < t_{zC}$, the system is stable under the introduction of the electron hopping between the two chains, contrary to the previous system without a magnetic field. Thus, when the magnetic flux of $\phi_n = 1/2$ is exerted, the system prefers to the condition for the hopping parameters $t_z$ and $t_y$. Physically speaking, this condition means that the double chains cannot collapse into the system that the plaquette has no area but must be stabilized by keeping a finite area of the plaquette [Fig. 16(a)]. And for the opposite case of $t_z > t_{zC}$, the system is stable to be well separated between the chains [Fig. 16(b)], since $r = 0$ (i.e. $t_y = 0$) at the ground state. Thus, in this case the electron hopping between chains is suppressed by the presence of the magnetic flux of $\phi_n = 1/2$.

6. The Quantum Ladder with a Staggered Magnetic Flux of $\phi_n = (-1)^{n+1} \phi_0$

6.1. Electronic structure

Let us consider the quantum ladder with a staggered magnetic flux [Fig. 6(a)]. To do so, we assume the staggered magnetic flux:

$$\theta_n = (-1)^n \theta_0 .$$  \hspace{1cm} (6.1)

From this using the definition of $\phi_n$ [Eq. (2.2)], we obtain the staggered magnetic flux:

$$\phi_n = (-1)^{n+1} \phi_0, \quad \phi_0 = \frac{\theta_0}{\pi} .$$  \hspace{1cm} (6.2)

This gives the inter-chain hopping potential:

$$V_n = e^{(-1)^n \theta_0 r} .$$  \hspace{1cm} (6.3)

This system is equivalent to not the staggered planar spin system but the tilted planar spin system [Fig. 6(b)] with Eq. (2.13).

Applying it to Eq. (3.4), we have two matrices $M_1$ and $M_2$:

$$M_1 = \begin{pmatrix}
E/t_z & -1 & -e^{-i\theta_0 r} & 0 \\
1 & 0 & 0 & 0 \\
-e^{i\theta_0 r} & 0 & E/t_z & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},$$  \hspace{1cm} (6.4)

$$M_2 = \begin{pmatrix}
E/t_z & -1 & -e^{i\theta_0 r} & 0 \\
1 & 0 & 0 & 0 \\
-e^{-i\theta_0 r} & 0 & E/t_z & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}.$$
Multiplying these we get
\[
M \equiv M_2M_1 = \begin{pmatrix}
E^2/t_z^2 + r^2e^{i2\theta_0} - 1 & E/t_z & 2r\cos\theta_0 E/t_z & re^{i\theta_0} \\
-E/t_z & 1 & -r e^{-i\theta_0} & 0 \\
2r\cos\theta_0 E/t_z & r e^{-i\theta_0} & E^2/t_z^2 + r^2 e^{-i2\theta_0} - 1 & E/t_z \\
-r e^{i\theta_0} & 0 & -E/t_z & 1
\end{pmatrix}.
\] (6.5)

Since \(M\) satisfies the condition (3.12), we can follow the scheme to obtain the energy bands. Calculating \(\text{Tr} \, M\) and \(\text{Tr}(M^2)\), we obtain
\[
\text{Tr} \, M = 2 \left( \frac{E^2}{t_z^2} + r^2 \cos 2\theta_0 - 2 \right),
\] (6.6)
\[
\text{Tr}(M^2) = 2 \left( \frac{E^4}{t_z^2} - 2A \frac{E^2}{t_z^2} + B \right),
\] (6.7)
\[
A = r^2(2\cos 2\theta_0 + 1) - 2,
\] (6.8)
\[
B = r^4 \cos 4\theta_0 - 4r^2 \cos 2\theta_0 + 2.
\] (6.9)

Substituting these into Eq. (3.21), we now get
\[
D = 2\text{Tr}(M^2) - (\text{Tr} \, M)^2 + 8 = 16r^2 \cos^2 \theta_0 \left( \frac{E^2}{t_z^2} - r^2 \sin^2 \theta_0 \right).
\] (6.10)

Hence, we obtain from Eq. (3.20)
\[
x_\pm = \frac{E^2}{t_z^2} + r^2 \cos 2\theta_0 - 2 \pm 2r \cos \theta_0 \sqrt{\frac{E^2}{t_z^2} - r^2 \sin^2 \theta_0},
\] (6.11)
where \(x_\pm = 2\cos 2k\) from the band condition for both channels.

Let us solve Eq. (6.11) for \(E\). First, we rewrite it as
\[
2 + 2\cos 2k - \frac{E^2}{t_z^2} - r^2 \cos 2\theta_0 = \pm 2r \cos \theta_0 \sqrt{\frac{E^2}{t_z^2} - r^2 \sin^2 \theta_0}.
\]

Squaring both sides, we get
\[
\left( 4\cos^2 k - \frac{E^2}{t_z^2} - r^2 \cos 2\theta_0 \right)^2 = 4r^2 \cos^2 \theta_0 \left( \frac{E^2}{t_z^2} - r^2 \sin^2 \theta_0 \right).
\]
Expand it with respect to \(E\), and we get
\[
\frac{E^4}{t_z^2} - 2(r^2 + 4 \cos^2 k) \frac{E^2}{t_z^2} + r^4 + 16 \cos^4 k - 8r^2 \cos 2\theta_0 \cos^2 k = 0.
\] (6.12)
Solving the above for $E^2/t_x^2$, we find
\[
\left( \frac{E^{(\pm)}}{t_x} \right)^2 = r^2 + 4 \cos^2 k \pm \sqrt{(r^2 + 4 \cos^2 k)^2 - r^4 - 16 \cos^4 k + 8r^2 \cos \theta_0 \cos^2 k}
\]
\[
= r^2 + 4 \cos^2 k \pm 4r \cos \theta_0 \cos k.
\]
Solving it once again for $E^{(\pm)}$, we finally obtain the following energy bands
\[
E^{(+)}_\pm(k) = \pm t_x \sqrt{r^2 + 4 \cos^2 k + 4r \cos \theta_0 \cos k},
\]
\[
E^{(-)}_\pm(k) = \pm t_x \sqrt{r^2 + 4 \cos^2 k - 4r \cos \theta_0 \cos k},
\]
where $E^{(+)}_\pm(k)$ and $E^{(-)}_\pm(k)$ stand for the upper and lower energy bands in channels $\pm$, respectively.

We now observe the following: (a) The spectrum is symmetric around $E = 0$. (b) If $r = 0$, then the above spectrum recovers that of the single chain, $E = -2t_x \cos k$. (c) If $r \neq 0$ and $\theta_0 = 0$, then it recovers that for no magnetic case [Eq. (4.4)], where the Brillouin zone is reduced to a half the original one (i.e. $-\pi/2 \leq k \leq \pi/2$) (Fig. 17). (d) If $r \neq 0$ and $\theta_0 = \pi/2$, then it coincides with that for

---

Fig. 17. Energy bands for the quantum ladder without a staggered magnetic flux, $\theta_0 = 0$. The case of $t_z = 1$ is shown with $r$ ranging from 0 to 3. The vertical axis means the energy $E$ and the horizontal axis the wavevector $k$ in units of $\pi$. The horizontal plane means the position of the Fermi energy (i.e. $E_F = 0$).
the uniform magnetic flux of $\phi_n = 1/2$ [Eq. (5.6)]. But in this case, spins are
aligned parallel to the $x$-direction (Fig. 18), which is different from the staggered
planar spins where spins are aligned parallel to the $z$-direction [Fig. 13(b)]. Why
is the spectrum of this case the same as that of the uniform magnetic flux case of
$\phi_n = 1/2$? The reason is that we can rotate all spins globally in the spin space
to the configuration of spins in the previous case, and vice versa. Hence, these two
cases must show the same spectrum. (e) If we assume $r \neq 0$ and $0 \leq \theta_0 \leq \pi/2$ such
that $\cos \theta_0 > 0$, then $E_+^{(+)}(k) > E_+^{(-)}(k) \geq 0 \geq E_-^{(-)}(k) \geq E_-^{(+)}(k)$ with respect to
$k$. Let us denote by $\Delta_0$ the difference between the bottom of $E_+^{(-)}$ and the top of
$E_-^{(-)}$, which measures the magnitude of a band gap at the center of the spectrum.
From Eq. (6.14b) together with the condition $(\partial E^{(-)}_{\pm}(k)/\partial k)|_{k=k_0} = 0$,
we find that there is a minimum (maximum) energy of $E_-^{(+)}(k)[E_-^{(-)}(k)]$ at a wave-
vector $k = k_0$, which is given by

$$k_0 = \begin{cases} \cos^{-1} \left( \frac{r \cos \theta_0}{2} \right) & \text{for } 0 \leq r \leq \frac{2}{\cos \theta_0}, \\
0 & \text{for } \frac{2}{\cos \theta_0} \leq r. \end{cases}$$

(6.15)

Fig. 18. The staggered planar spin chain. The spins are aligned to the $x$-direction.

Fig. 19. The typical energy bands for the quantum ladder with a staggered magnetic flux of
$\theta_0 = \pi/4$. It is shown for $r = 1$. There is a minimum of the upper energy band ($E_+^{(-)}$) at
$k = \pm k_0$, while there is a maximum of the lower energy band ($E_-^{(-)}$) at $k = \pm k_0$. This provides
the band gap $\Delta_0 = E_+^{(-)}(k_0) - E_-^{(-)}(k_0) = 2t_y \sin \theta_0$. 
Substituting the above into Eq. (6.14) we find the band gap at the wavevector $k = k_0$:

$$\Delta_0 = E_+^{(-)}(k_0) - E_-^{(-)}(k_0) = 2t_y \sin \theta_0 \leq 2t_y. \quad (6.16)$$

The typical energy bands are shown in Fig. 19 for $r = 1$ and $\theta_0 = \pi/4$ and the energy bands with respect to $r$ are also shown in Fig. 20 taking the value of $\theta_0 = \pi/4$.

Fig. 20. Energy bands for the quantum ladder with a staggered magnetic flux of $\theta_0 = \pi/4$. The case of $t_x = 1$ is shown with $r$ ranging from 0 to 4. The vertical axis means the energy $E$ and the horizontal axis the wavevector $k$ in units of $\pi$. The horizontal plane means the position of the Fermi energy (i.e. $E_F = 0$).

### 6.2. The ground state energy of the system

The magnetic energy $E_m$ of the system is given by

$$E_m = t_x^2 t_y^2 \sum_n (1 - \cos(\theta_{n+1} - \theta_n)) = t_x^2 t_y^2 \sum_n (1 - \cos(2(-1)^n \theta_0))$$

$$= N_x t_x^2 t_y^2 (1 - \cos 2\theta_0). \quad (6.17)$$
Let us calculate the ground state energy $E_g$ of the system, which is given by

$$E_g = 2 \sum_{k, \text{occ.}} E_{-}^{(+)}(k) + 2 \sum_{k, \text{occ.}} E_{-}^{(-)}(k) + E_m$$

$$= \frac{2N_s}{\pi} \int_{-\pi/2}^{\pi/2} E_{-}^{(+)}(k) dk$$

$$+ \frac{2N_s}{\pi} \int_{-\pi/2}^{\pi/2} E_{-}^{(-)}(k) dk + E_m,$$  \hspace{1cm} (6.18)

where the factor 2 in front of the integrals comes from spin degeneracy. Substituting Eq. (6.14) into Eq. (6.18), we obtain

$$E_g \equiv N_s \varepsilon_g(r) = -\frac{4N_s t_x}{\pi} \int_{0}^{\pi/2} \sqrt{r^2 + 4 \cos^2 k + 4r \cos \theta \cos k} dk$$

$$- \frac{4N_s t_x}{\pi} \int_{0}^{\pi/2} \sqrt{r^2 + 4 \cos^2 k - 4r \cos \theta \cos k} dk$$

$$+ N_s t_x^2 r^2 (1 - \cos 2\theta).$$  \hspace{1cm} (6.19)

Thus, we obtain the ground state energy per site:

$$\varepsilon_g(r)/t_x = -\frac{4}{\pi} \int_{0}^{\pi/2} \sqrt{r^2 + 4 \cos^2 k + 4r \cos \theta \cos k} dk$$

$$- \frac{4}{\pi} \int_{0}^{\pi/2} \sqrt{r^2 + 4 \cos^2 k - 4r \cos \theta \cos k} dk$$

$$+ t_x^2 r^2 (1 - \cos 2\theta).$$  \hspace{1cm} (6.20)

The above ground state energy is not simply represented by the elliptic integral as before, but we can calculate it numerically. Equation (6.20) is shown in Figs. 21(a) and (b) for (a) $t_x < t_{xc}$ and (b) $t_x > t_{xc}$, respectively, changing the value $r$. In case (a), we are able to see the double well structure with a minimum at $r = \pm r_c$ as well. On the other hand, in case (b) we see the single well structure with a minimum at $r = 0$. This situation is also very reminiscent of that in polyacetylene.\textsuperscript{16-18} Therefore, as was mentioned before for the quantum ladder with a uniform magnetic flux of $\phi_n = 1/2$, in case (a) the system is stable with a finite separation between the two chains so that each plaquette has a finite area. But in case (b) the system is stable when the two chains are well separated in the distance. Thus, again we can claim that there is a transition between the case (a) and the case (b) changing the value of $t_x$. 
Fig. 21. The ground state energy $E_g$ of the quantum ladder with a staggered magnetic flux of $	heta_0 = \pi/4$. (a) $t_\phi < t_{\phi C}$; (b) $t_\phi > t_{\phi C}$. 
7. Conclusion

In conclusion we have formulated a tight-binding model for the calculation of the electronic structure of the quantum ladder under a magnetic field. To demonstrate how the theory works, we have applied it to obtaining the electronic structure and the ground state energy for the three types of the quantum ladders without magnetic flux of $\phi_n = 0$, with the uniform magnetic flux of $\phi_n = 1/2$, and with the staggered magnetic flux of $\phi_n = (-1)^{n+1} \phi_0$.

We found the following: In the electronic spectrum, as the anisotropy parameter $r$ is increased, there are a metal–semimetal transition at $r = 0$ and a semimetal–semiconductor transition at $r = 2$ in the first case, and there are metal–semiconductor transitions at $r = 0$ in the second and the third cases. Physically speaking, it is quite important because even such simple systems can show a new category of the M–I transitions due to the hopping anisotropy of the system. It is distinct from the usual concept of the crossover between one-dimensional and two-dimensional characters of the system, where the spectrum changes from one-dimensional energy bands to two-dimensional ones. Because in our case the spectrum keeps the same one-dimensional character between the transition except that the energy bands change due to the formation of a quantum ladder structure. On the other hand, it was found that as $r$ is increased, the ground state energy of the first system diverges such that there is an instability of the system, while those of the second and third systems may have the single or double well structures such that the system is stable at some value of $r = r_c$. It has been pointed out that this situation is very reminiscent of that in polyacetylene.\(^{16-18}\)

Finally, we would like to emphasize that the formalism, that is proposed in this paper, itself [Eq. (3.20)] is very general enough to treat the quantum ladder with a very large nonperiodic unit cell due to the aperiodic magnetic flux $\phi_n$, since $q$ in Eq. (3.9) can be arbitrarily large. Although these cases were not performed in this paper, the theory works even for these quantum ladder systems under a quasiperiodic and random magnetic fluxes as well. Thus, we believe that this type of theory may be useful in analyzing the quantum ladder under a very complicated and nonperiodic magnetic flux as well as a random one. This direction will be very interesting but we leave these for the future.

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References