Equivalence between the Nielsen and the scaling transformations in one-dimensional quasiperiodic systems

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(Received 13 November 1992; accepted for publication 9 March 1993)

An equivalence between the Nielsen transformations in mathematics and the scaling transformations in quasiperiodic physics in one dimension is shown herein. By recognizing this equivalence one becomes able to obtain the trace map for the quasiperiodic lattices constructed by an arbitrary number of atoms in the chain, where the atoms are symbolized by letters. In this approach the scaling transformations are regarded as the automorphisms of the set of letters, which induce the trace map as the inner automorphisms.

I. INTRODUCTION

There has been much interest on a mathematical study of Schrödinger operators with an arbitrary deterministic potential sequence since the discovery of quasicrystals and their one-dimensional modeling. On the way from periodic to random sequences, investigation has been focused on quasiperiodic systems (such as the Fibonacci chain and its generalized ones) and on aperiodic systems (such as the Thue–Morse sequence). These sequences belong to a class of sequences generated by a substitution of two letters.

Some generalizations of the sequences including more than two letters have been performed by Luck, and Dulea, Johansson, and Riklund for the Rudin–Shapiro sequence, by Aubry, Godrèche, and Luck for the circle sequence, by Ali and Gumbs, and Iguchi for the ternary quasiperiodic sequences.

Towards this direction some doubts have been addressed by Luck on whether or not the trace map exists under any substitution of more than four letters, because if it exists, one is able to calculate the physical properties in the systems. Kolář and Nori partially answered this question, and Avishai and Berend have given a more precise discussion.

In this article I would like to show that the above systems fall into a class of systems that are generated by an old mathematical concept—the Nielsen transformations. This concept gives us a unified picture of our problem. In this approach the deterministic substitutions (i.e., the scaling transformations) can be regarded as the automorphisms of a free group generated by the letters. And the trace maps are induced from these automorphisms as inner automorphisms.

The organization of the article is the following. In Sec. II, I shall show the equivalence between the Nielsen transformations in mathematics and the scaling transformations in quasiperiodic physics in one dimension. And it will be shown that the Nielsen transformations are generators for the scaling transformations that are regarded as automorphisms of a free group generated by n letters. In Sec. III, the concept of the canonical projection is introduced and it will be shown that the projection is equivalent to the commutator quotient group in mathematics. In Sec. IV, the trace maps are given as the inner automorphisms of the free group. In Sec. V, I shall conclude what has been done.

II. THE NIELSEN TRANSFORMATIONS

Denote the species of atoms by a set of n letters \( \Lambda_n=\{a,b,c,...,z\} \). This set forms a basis for construction of a lattice. And define its conjugate as \( \Lambda_n^{-1}=\{a^{-1},b^{-1},c^{-1},...,z^{-1}\} \) such that \( aa^{-1}=a^{-1}a=1 \), and so forth. Then one has a set of \( 2n+1 \) letters.
I remark the following. Physicists are concerned with the one-dimensional lattice problem to understand its physical properties. In this case, the letters are thought of as the $2 \times 2$ matrices belonging to $G=SL(2,\mathbb{C})$ defined by physical models such as the Schrödinger equation, the Maxwell equation, etc. For this purpose we denote the matrix-valued letters by their capital letters $A,B,C,...,Z$ corresponding to the $n$ letters $a,b,c,...,z$. And a string of letters (i.e., a word) is regarded as the unit cell for an infinite chain, say $L(a,b,c,...,z)$.

Denote the $n$-tuple of the initial set of the unit cells by

$$U_0 = (L_{n-1}(a,b,c,...,z), L_{n-2}(a,b,c,...,z),..., L_0(a,b,c,...,z)).$$  \hspace{1cm} (2.2)

Then inflation-deflation (i.e., the scaling transformation or the renormalization group transformation) can be thought of as a substitution scheme of letters into them: If $n$ strings of words: $W_1(a,b,c,...,z), W_2(a,b,c,...,z),..., W_n(a,b,c,...,z)$ are set, then a substitution scheme $S$ that inflates (or deflates) the lattice from one generation to another is defined as follows:

$$S: \begin{cases} 
  a \rightarrow W_1(a,b,c,...,z) \\
  b \rightarrow W_2(a,b,c,...,z) \\
  \vdots \\
  z \rightarrow W_n(a,b,c,...,z).
\end{cases}$$  \hspace{1cm} (2.3)

For example, for a Fibonacci lattice case, the inflation scheme is given by the substitution $a \rightarrow W_1(a,b) = ab, b \rightarrow W_2(a,b) = a$. For the Thue-Morse lattice case, the inflation scheme is given by the substitution $a \rightarrow W_1(a,b) = ab, b \rightarrow W_2(a,b) = ba$. For a quasiperiodic lattice case such as the generalization of the Fibonacci lattice, the substitution is $a \rightarrow W_1(a,b) = a^n b, b \rightarrow W_2(a,b) = a$, where $a^n = a a \cdots a$ and $n$ can be an arbitrary positive integer at each step of the inflation.

The above substitution acts on the $n$-tuple such that the set of unit cells is transformed to obtain a new set of unit cells that consist of the inflated numbers of letters. Thus one obtains a mapping

$$S: U_0 = (L_{n-1}, L_{n-2},..., L_0) \rightarrow U_1 = (L_{n-1}', L_{n-2}',..., L_0'),$$  \hspace{1cm} (2.4)

i.e., $U_1 = SU_0$. Recursively doing this mapping one gets

$$U_{k+1} = SU_k.$$  \hspace{1cm} (2.5)

Since the substitution can be arbitrary at each stage of the scaling transformations, one can replace $S$ by $S_k$ such that

$$U_{k+1} = S_k U_k.$$  \hspace{1cm} (2.6)

Mathematically speaking, this type of transformation is equivalent to a generator of an automorphism on the free group $F_n$ constructed by $A_n$. There is a mathematical theorem by Nielsen:

**Nielsen’s Theorem:** Any substitution is generated by a combination of the five generators: inversion (J), exchange (X), cyclic permutation (P), and left and right multiplications (L and R) (see Table 1). By this theorem any substitution $S_k$ can be decomposed into a string of generators in five classes $J, X, P, L$, and $R$, where $J^2 = X^2 = P^n = 1$ (identity), and call a free group generated by them $N_n$.

For example, if the system is inflating as
TABLE I. The Nielsen transformations for \( n \) letters. This shows only one example for inversion, exchange, left and right multiplications since there are the \( n \) generators for \( J \), and the \( n(n-1)/2 \) generators for exchange \( X \), left and right multiplications \( L \) and \( R \).

| \( \text{J}_1 \) | \( A^{-1} \) | \( B \) | \( C \) | \( D \) | \ldots | \( Z \) |
| \( \text{X}_{12} \) | \( B \) | \( A \) | \( C \) | \( D \) | \ldots | \( Z \) |
| \( \text{P} \) | \( Z \) | \( A \) | \( B \) | \( C \) | \ldots | \( Y \) |
| \( \text{L}_{12} \) | \( BA \) | \( B \) | \( C \) | \( D \) | \ldots | \( Z \) |
| \( \text{R}_{12} \) | \( AB \) | \( B \) | \( C \) | \( D \) | \ldots | \( Z \) |

\[ z \rightarrow y \rightarrow x \rightarrow \cdots \rightarrow b \rightarrow a \rightarrow abc \cdots z \rightarrow \cdots, \tag{2.7} \]

then one can set the initial set as

\[ L_0(a,b,c,\ldots,z) = z, \]

\[ L_1(a,b,c,\ldots,z) = y, \]

\[ \vdots \]

\[ L_{n-1}(a,b,c,\ldots,z) = a, \]

\[ L_n(a,b,c,\ldots,z) = abc \cdots z, \tag{2.8} \]

and so forth. This scaling transformation is represented by a string of the five set of the generators such as

\[ S_k = \text{PR}_1n\text{R}_2n\text{R}_3n \cdots \text{R}_{n-1}n. \tag{2.9} \]

Similarly for the sequence of inflation

\[ z \rightarrow y \rightarrow x \rightarrow \cdots \rightarrow b \rightarrow a \rightarrow a^{m_1}b^{m_2}c^{m_3} \cdots y^{m_{n-1}}z \rightarrow \cdots, \tag{2.10} \]

one gets

\[ S_k = \text{PR}_1n^{m_1}\text{R}_2n^{m_2}\text{R}_3n^{m_3} \cdots \text{R}_{n-1}n^{m_{n-1}}, \tag{2.11} \]

where \( m_j \) are positive integers.

In physics a very similar approach has been independently introduced and is called the scaling group by the author to derive the trace map for arbitrary binary\(^6\) and ternary\(^12\) quasiperiodic lattices. I shall show later that these generators of automorphisms induce the canonical representation on words and the trace maps on the traces.

III. THE CANONICAL PROJECTION

The representation of a word is basically infinite-dimensional since the dimension of the free group is infinite. There is a natural projection of a word, which makes the representation finite-dimensional. I shall call this representation the canonical projection.\(^6,12\) By this projection the dimension becomes the total number of the distinct letters in words.

The canonical projection of words is defined as follows. If a string of a word such as \( W_j(a,b,c,\ldots,z) \) is taken into account, then we count the total numbers of \( a,b,c,\ldots,z \) in this word as \( n_{a_j}, n_{b_j}, n_{c_j}, \ldots, n_{z_j} \), respectively. Denote by \( \pi \) the canonical projection of the word \( W_j \) such...
that $\pi(W_j(a,b,c,\ldots,z)) = (n_{aj}, n_{bj}, n_{cj}, \ldots, n_{xj})$. If there are $n$ strings of the words: $W_1(a,b,c,\ldots,z), W_2(a,b,c,\ldots,z), \ldots, W_n(a,b,c,\ldots,z)$, then one can obtain their canonical projection by means of an $n \times n$ matrix

$$
\pi \left( \begin{array}{c} W_1(a,b,c,\ldots,z) \\
W_2(a,b,c,\ldots,z) \\
\vdots \\
W_n(a,b,c,\ldots,z) 
\end{array} \right) = 
\left( \begin{array}{c} n_{a1} n_{b1} n_{c1} \cdots n_{x1} \\
 n_{a2} n_{b2} n_{c2} \cdots n_{x2} \\
\vdots \\
 n_{an} n_{bn} n_{cn} \cdots n_{xn} 
\end{array} \right) \equiv \Pi.
$$

(3.1)

For example, for the Fibonacci lattice case, the canonical projection of the inflation is given by the following $2 \times 2$ matrix:

$$
\pi \left( \begin{array}{c} W_1(a,b) \\
W_2(a,b) 
\end{array} \right) = 
\left( \begin{array}{cc} 1 & 1 \\
1 & 0 
\end{array} \right).
$$

(3.2)

Similarly, for the Thue–Morse lattice case, it is given by

$$
\pi \left( \begin{array}{c} W_1(a,b) \\
W_2(a,b) 
\end{array} \right) = 
\left( \begin{array}{cc} 1 & 1 \\
1 & 1 
\end{array} \right)
$$

(3.3)

and for a quasiperiodic lattice case

$$
\pi \left( \begin{array}{c} W_1(a,b) \\
W_2(a,b) 
\end{array} \right) = 
\left( \begin{array}{cc} n & 1 \\
1 & 0 
\end{array} \right).
$$

(3.4)

In physics this was first introduced by Lu, Odagaki, and Birman.\textsuperscript{18} I now find that it is exactly equivalent to the \textit{commutator quotient groups} in mathematics.\textsuperscript{16}

The generators for the group of the canonical projection are obtained according to those for the Nielsen transformations $N_n$. For $J_1$, $X_{12}$, $P$, $L_{12}$, and $R_{12}$, they are given as follows:

$$
\pi(J_1) = 
\left( \begin{array}{cccccccc}
-1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 
\end{array} \right),
$$

(3.5)

$$
\pi(X_{12}) = 
\left( \begin{array}{cccccccc}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 
\end{array} \right),
$$

(3.6)

$$
\pi(P) = 
\left( \begin{array}{cccccccc}
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & 0 
\end{array} \right),
$$

(3.7)
Here it turns out that the above matrix representations for the generators are identical to the
generators of the unimodular group \( M_n \).\(^1\)

By a similar procedure of the canonical projection on the initial \( n \)-tuple \( U_0 = (L_{n-1}, L_{n-2}, \ldots, L_0) \) one obtains

\[
\begin{pmatrix}
L_{n-1}(a,b,c,\ldots,z) \\
L_{n-2}(a,b,c,\ldots,z) \\
\vdots \\
L_0(a,b,c,\ldots,z)
\end{pmatrix}
\begin{pmatrix}
A_{n-1} & B_{n-1} & C_{n-1} & \cdots & Z_{n-1} \\
A_{n-2} & B_{n-2} & C_{n-2} & \cdots & Z_{n-2} \\
\vdots \\
A_0 & B_0 & C_0 & \cdots & Z_0
\end{pmatrix} \equiv P_0,
\]

(3.9)

where \( A_j, B_j, \ldots, Z_j \) are the total numbers of \( a, b, \ldots, z \) in the unit cell \( L_j(a,b,c,\ldots,z) \) \((j=0,\ldots, n-1)\), respectively, while the total number of letters in the unit cell \( N_j \) is obviously given by

\[
N_j = A_j + B_j + \cdots + Z_j.
\]

(3.10)

Operating \( \Pi \) on \( P_0 \) provides \( P_1 = \Pi P_0 \). Recursively operating it, one then gets \( P_{k+1} = \Pi P_k \).

Thus, the canonical projection for the scaling transformation [Eq. (2.6)] is given by

\[
P_{k+1} = \Pi_k P_k,
\]

(3.11)

where since the scaling transformation is decomposed into a string of the generators in \( N_n \), so is \( \Pi_k \). Namely, \( \Pi_k \) is decomposed into a string of \( \tau(J_1), \tau(X_{12}), \tau(P), \tau(L_{12}), \) and \( \tau(R_{12}) \), etc.

For example, for the scaling transformations given by Eq. (2.7) and Eq. (2.10), one has the following:

\[
\Pi_k = \tau(PR_{1n}R_{2n}R_{3n}\cdots R_{n-1n}) = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix},
\]

(3.12)

\[
\Pi_k = \tau(PR_{1n}^{m_1}R_{2n}^{m_2}R_{3n}^{m_3}\cdots R_{n-1n}^{m_{n-1}}) = \begin{pmatrix}
m_1 & m_2 & \cdots & m_{n-1} & 1 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

(3.13)

From \( \Pi_k \) together with Eq. (3.11), the development of the total number of letters \( N_j \) is represented by
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\[ N_{j+n} = N_{j+n-1} + N_{j+n-2} + \cdots + N_{j+1} + N_j \]  
(3.14)

for Eq. (3.12), and

\[ N_{j+n} = m_{n-1}N_{j+n-1} + m_{n-2}N_{j+n-2} + \cdots + m_1N_{j+1} + N_j \]  
(3.15)

for Eq. (3.13) with the initial condition

\[ N_n = N_{n-1} = \cdots = N_1 = 1. \]  
(3.16)

The ratio \( N_{j+1}/N_j \) can converge to an irrational number that is one of the roots of an algebraic equation associated with Eqs. (3.14) and (3.15) as \( j \to \infty \). In this way one can recognize that the canonical projection works well and Eq. (3.11) is a generalized version of the continued fraction expansion for the systems to two letters to our problem.6,11,12,15,18

\section{IV. TRACE MAPS}

I am going to answer whether or not the trace map exists for any substitutional sequence generated by \( \Lambda_n \).

Denote the \( n \) traces by

\[ t_1 = \text{tr} A, \quad t_2 = \text{tr} B, \quad t_3 = \text{tr} C, \ldots, t_n = \text{tr} Z. \]  
(4.1)

And also denote the trace of \( AB \) by

\[ t_{12} = \text{tr}(AB) \]  
(4.2)

and likewise, we define

\[ t_{123} = \text{tr}(ABC) \]  
(4.3)

and so forth. The following are some symmetric properties of traces

\[ t_1 = \text{tr}(A^{-1}) = t_1; \quad \text{inversion symmetry}, \]  
(4.4)

\[ t_{12} = t_{21}; \quad \text{exchange symmetry}, \]  
(4.5)

\[ t_{123} = t_{231} = t_{312}; \quad \text{cyclic permutation symmetry}, \]  
(4.6)

etc.

There exist some identities which are sometimes called Fricke identities\textsuperscript{20,21}

\[ \text{tr}(A^{-1}B) = \text{tr} A \cdot \text{tr} B - \text{tr}(AB), \]  
(4.7)

\[ \text{tr}(A^{-1}B^{-1}AB) = (\text{tr} A)^2 + (\text{tr} B)^2 + (\text{tr}(AB))^2 - \text{tr} A \cdot \text{tr} B \cdot \text{tr}(AB) - 2. \]  
(4.8)

One then is able to rewrite the above equations as

\[ t_{12} = \text{tr}(A^{-1}B) = t_1t_2 - t_{12}, \]  
(4.9)

\[ A_{12} = \text{tr}(A^{-1}B^{-1}AB) = t_1^2 + t_2^2 + t_{12}^2 - t_1t_2t_{12} - 2. \]  
(4.10)
The above two identities hold for any combination of the two matrices in the set of the transfer matrices. It is worth mentioning that Eq. (4.10) is equivalent to the invariant surface first found by Kohmoto, Kadanoff, and Tang: If we parametrize $x=t_1/2$, $y=t_2/2$, and $z=t_12/2$, then

$$I_{12} = (A_{12} - 2)/4 = x^2 + y^2 + z^2 - 2xyz - 1. \quad (4.11)$$

Furthermore, we have the following identities:

$$p = t_{13} + t_{123} = t_{13} + t_{132} + t_{12} + t_{123}, \quad (4.12)$$

$$q = t_{123} + t_{132} = t_1^2 + t_2^2 + t_3^2 + t_12^2 + t_23^2 + t_31^2 + t_123 - t_1t_2t_3 - t_{13} - 4. \quad (4.13)$$

From these, $t_{123}$ and $t_{132}$ are the roots of a quadratic equation

$$p^2 - pt + q = 0, \quad (4.14)$$

respectively. In this way a trace of the three different matrices is represented in terms of the six traces: $t_1$, $t_2$, $t_3$, $t_{12}$, $t_{23}$, and $t_{13}$. This eventually provides the minimal dimension of the trace map for the ternary quasiperiodic systems.

Let us first consider the actions of the generators of the Nielsen transformations $N_n$ on the $n(=nC_1)$ traces

$$t_1, t_2, ..., t_n. \quad (4.15)$$

One finds that the left or right multiplication relates the $n$ traces to the $n(n-1)/2(=nC_2)$ traces of two matrices

$$t_{12}, t_{23}, ..., t_{n-1n}. \quad (4.16)$$

For example, if one substitutes $AB$ into $t_i = \text{tr}(A)$, then it becomes $t_{12}$. The result is summarized in Table II.

Second, one has to take into account the actions of the five generators on the $n(n-1)/2$ traces as well. The action of inversion $J_i$ is the following: If one substitutes $A^{-1}$ into $A$ in $t_{12} = \text{tr}(AB)$, then one gets $t_{12} = \text{tr}(A^{-1}B)$. By the Fricke identity, Eq. (4.9), it becomes $t_{12} = t_1t_2 - t_{12}$. Let us consider the action of the left or right multiplication. If one substitutes $AB$
TABLE III. The actions of the generators for the Nielsen transformations on the traces. The case of \( n=4 \) is shown.

| \( J_1 \) | \( t_{134} - t_{123} \) | \( t_{134} - t_{123} \) | \( t_{134} - t_{123} \) | \( t_{134} - t_{123} \) |
| \( X_{12} \) | \( t_{123} \) | \( t_{123} \) | \( t_{123} \) | \( t_{123} \) |
| \( P \) | \( t_{124} \) | \( t_{124} \) | \( t_{124} \) | \( t_{124} \) |
| \( L_{12} \) | \( t_{1234} - t_{134} \) | \( t_{1234} - t_{134} \) | \( t_{1234} - t_{134} \) | \( t_{1234} - t_{134} \) |
| \( R_{12} \) | \( t_{1234} - t_{134} \) | \( t_{1234} - t_{134} \) | \( t_{1234} - t_{134} \) | \( t_{1234} - t_{134} \) |

\[
t_{1234} = -t_{1234} + (t_{12} - t_{13})t_{34} + t_{13}t_{24} + t_{23}t_{14}
\]

into \( \mathcal{A} \) in \( t_{12} = \text{tr}(AB) \) by left or right multiplication, then it becomes \( t_{122} = t_{212} = \text{tr}(ABB) \). Since \( B^2 = \text{tr}(B)B - I \), one obtains \( \text{tr}(ABB) = \text{tr}(B)\text{tr}(AB) - \text{tr}(A) = t_{212} - t_{12} \). Similarly, if one substitutes \( BA \) (or \( AB \)) into \( \mathcal{A} \) in \( t_{13} = \text{tr}(AC) \) by left (or right) multiplication, then it becomes \( t_{132} = \text{tr}(BAC) \) [or \( t_{123} = \text{tr}(ABC) \)], where the relationship between \( t_{132} \) and \( t_{123} \) obeys the Fricke identity, Eqs. (4.12)-(4.13). In this way the actions of the generators relate the \( n(n-1)/2 \) traces to \( n(n-1)(n-2)/6(=nC_3) \) traces

\[
t_{123} \cdots t_{n-2} - n = t_{n}. \quad (4.17)
\]

This result is also summarized in Table II.

Finally one finds that \( 2^n - 1 (=\_nC_1 + _nC_2 + \cdots + _nC_n) \) traces must be treated under the actions of the generators. This result is summarized in Table III, where the \( n=4 \) case is demonstrated.

I note the following: In order to derive the trace maps under the actions of the generators of the Nielsen transformations, the Fricke identities have to be fully used, together with Horowitz's theorem.

**Horowitz's Theorem:** If \( u \) is an arbitrary element of \( F_n \), then the trace of \( u \) can be expressed as a polynomial

\[
\text{tr} \, u = p(t_{12}, t_{13}, \ldots, t_{n-1}; t_{12}, t_{23}, \ldots, t_{n-1}; t_{34}, \ldots, t_{n-2}; t_{12}, \ldots, t_{n-1}; t_{12}, \ldots, t_{n-1}),
\]

(4.18)

with integer coefficients in the \( 2^n - 1 \) traces. This is proved by using the following lemma:

**Lemma:** Take the \( s \) generators \( g_1, g_2, \ldots, g_s (s < n) \) in \( F_n \). Then there is the following identity:

\[
\text{tr}(g_1g_2 \cdots g_{s-1}g_s) = \text{tr}(g_1g_2 \cdots g_{s-2}) \left[ \text{tr}(g_{s-1}g_s) - \text{tr}(g_s) \right] \text{tr}(g_{s-1}) + \text{tr}(g_{s-1}) \text{tr}(g_1g_2 \cdots g_{s-2}g_s) + \text{tr}(g_s) \text{tr}(g_1g_2 \cdots g_{s-2}g_s) - \text{tr}(g_1g_2 \cdots g_{s-1}g_s) \text{tr}(g_{s-1}g_s - 1).
\]

(4.19)

The proof of this lemma is easy. One may just substitute \( A = g_1g_2 \cdots g_{s-2}, B = g_{s-1}, \) and \( C = g_s \) into the Fricke identity, Eq. (4.12).

For example, consider \( t_{2134} \). From the above lemma, one finds an identity

\[
t_{2134} = -t_{1234} + (t_{12} - t_{13})t_{34} + t_{13}t_{24} + t_{23}t_{14}.
\]

(4.20)
where the expression $B(12;34)$ is obviously symmetric under the exchange between 1 and 2, namely, $B(12;34) = B(21;34)$. Repeating the above interchange three times, $t_{1234} = -t_{1234} + B(12;34) = -t_{2341} + B(23;41) - B(24;13) - B(12;34)$.

Therefore, one gets

$$2 \cdot t_{1234} = B(23;41) - B(24;13) + B(12;34)$$

$$= (t_{23} - t_{24})t_{14} + t_{24}t_{13} + t_{21}t_{14} - [(t_{24} - t_{2}t_{4})t_{13} + t_{2}t_{413} + t_{4}t_{213}]$$

$$+ (t_{12} - t_{1}t_{2})t_{34} + t_{1}t_{234} + t_{2}t_{134}.$$ (4.22)

The above identity agrees with the sixth equation in the theorem by Whittemore. In this way a trace of four letters is represented in terms of the traces of less than three letters.

Denote by $\Psi_0$ the initial set of all the $2^n - 1$ traces. Operating the scaling transformation $S_0$, one gets $\Psi_1 = S_0 \Psi_0$. Recursively operating this, one gets the trace map for our system as

$$\Psi_{k+1} = S_k \Psi_k.$$ (4.23)

Since the arbitrary substitution is represented by a string of the five generators, the dimension of the trace map is invariant under the scaling transformation and apparently it is $2^n - 1$. This result agrees with that by Avishai and Berend, but it may not be minimal since there are the Fricke identities as constraints.

In this way the trace map is induced from the automorphisms of the free group $F_n$. This is due to the following fact: Let $g$ be an element of $G = SL(2,\mathbb{C})$. Let $t$ be an element of $N_n$. Then one proves that $N_n$ commutes with the inner automorphisms of $G$: For $(A,B,C,...,Z) \in G \times G \times G \times \cdots \times G$,

$$t[g(A,B,C,...,Z)g^{-1}] = g[t(A,B,C,...,Z)]g^{-1}.$$ (4.24)

This implies that $N_n$ induces a symmetry group $T_n$ called the trace map on the invariant $(A,B,C,...,Z)$ under inner automorphisms.

An important problem left here is to answer whether or not the dimension is minimal. Since all the traces more than three letters are represented by the $n$ traces of one letter and the $n(n-1)/2$ traces of two letters, there appears no extra dimensions under the automorphisms. Thus, one can conclude that the minimal dimension is $n(n+1)/2$. For $n = 1, 2, 3$, and 4, it is one, three, six, and ten, respectively. This agrees with known results for $n = 2$ (Refs. 3, 5, 6, and 13) and for $n = 3$ (Refs. 11 and 12) and gives new results for $n \geq 4$.

V. CONCLUSION

In conclusion it is shown that the scaling transformations in the quasiperiodic and aperiodic physics in one dimension is equivalent to an old mathematical concept—the Nielsen transformations—which are automorphisms of a free group $F_n$ generated by $n$ letters. I have shown that the automorphisms induce both the canonical projection on words (which can be regarded as a generalized version of a continued fraction expansion in the systems of two letters to our systems) and the trace map on the traces (which is regarded as the inner automorphisms). I finally summarize the correspondence between mathematics and physics in Table IV.
TABLE IV. The terminological correspondence between physics and mathematics. This shows that there is perfect equivalence between them.

<table>
<thead>
<tr>
<th>Physics</th>
<th>Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atoms</td>
<td>Generators of a free group</td>
</tr>
<tr>
<td>Unit cells</td>
<td>Words</td>
</tr>
<tr>
<td>A set of all types of unit cells</td>
<td>A free group generated by letters</td>
</tr>
<tr>
<td>Substitutions of letters</td>
<td>Nielsen transformations</td>
</tr>
<tr>
<td>Scaling transformations</td>
<td>Automorphisms of a free group</td>
</tr>
<tr>
<td>Canonical projection</td>
<td>Commutator quotient group</td>
</tr>
<tr>
<td>Trace map</td>
<td>Inner automorphisms</td>
</tr>
</tbody>
</table>

ACKNOWLEDGMENTS

I would like to thank Y. Avishai and D. Berend for giving me a preprint of their work prior to publication.

4. We would like to refer to another method in the following articles that treat not the trace map but the decimation technique of the renormalization group. Q. Niu and F. Nori, Phys. Rev. Lett. 57, 2057 (1986); Phys. Rev. B 42, 10329 (1990).
17. There are many other relations on the generators. See Ref. 16.

J. Math. Phys., Vol. 34, No. 8, August 1993